A GENERALIZATION OF FUBINI'S THEOREM FOR BANACH ALGEBRA-VALUED MEASURES

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ABSTRACT. The present paper gives a generalization of Fubini's theorem when the function f and the vector measures μ_1 and μ_2 of bounded variation assume values in a Banach algebra.

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Fubini's theorem for Bochner integrals with values in a Banach space has been known for a long time. (See Hille and Phillips [5]). The object of the present work is to treat a generalization of this theorem when the function f and the vector measures μ_1 and μ_2 of bounded variation assume values in a Banach algebra. When the Banach algebra is commutative and the range of f is bounded, this generalization reduces to the theorem of Fubini in this case.

§1. PRELIMINARIES

In this section we give some definitions and results from the literature, on the theory of integration with respect to Banach algebra-valued measures of bounded variation.

$X \neq \{0\}$ will denote in the sequel a Banach algebra (real or complex) with norm $\|.\|$. X is not assumed to have an identity. Unless otherwise mentioned, X is not commutative.

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Let Σ be a σ-ring of subsets of a set $\Omega \neq \phi$. $\mu:\Sigma \to X$ is called a measure if μ is countably additive in Σ, with respect to the norm topology of X. μ is called a measure of bounded variation when $\sup_{E\in\Sigma} |\mu|(E) < \infty$, where $|\mu|$ denotes the variation of μ. As $|\mu|$ is countably additive in the σ-ring Σ, μ is of bounded variation if and only if $|\mu|(E) < \infty$ for all $E \in \Sigma$.

If $\nu:\Sigma \to [0,\infty]$ is a positive measure, $\nu^*(E)=\inf\{\nu(F): E\subset F\in\Sigma\}$ is an outer measure on the hereditary σ-ring $H(\Sigma)$ generated by Σ. Let $M_\nu = \{E \in H(\Sigma): E$ ν^*-measurable$\}$. Let $\tau(\nu)=\{E\subset\Omega: E\cap A\in M_\nu$ for every $A \in M_\nu\}$. The members of $\tau(\nu)$ are called ν-measurable sets. It is known that $\tau(\nu)$ is a σ-algebra containing M_ν and hence containing Σ. (Vide. p. 70 [2]).

The set function $\nu^*:\tau(\nu)\to[0,\infty]$, defined by

$$\nu^*(E) = \sup_{\substack{A\subset E \\ A\in M_\nu}} \nu^*(A)$$

is a positive measure and extends ν^* from M_ν to $\tau(\nu)$. The sets $E \in \tau(\nu)$ with $\nu^*(E) = 0$ are called ν-negligible. The notion of almost everywhere with respect to ν is defined en terms of ν-negligible. Also we shall denote ν^* by ν on $\tau(\nu)$.

A function $f:\Omega \to X$ is called Σ-simple if it admits a representation of the form

$$f(w) = \sum_{i=1}^{n} x_i \chi_{E_i}(w)$$

where $x_i \in X$, $E_i \in \Sigma$, $i = 1,2,.n$. It is true that

$$N(f) = \{w: f(w) \neq 0\} \in \Sigma.$$

(Vide Remarks p. 83, [2])

DEFINITION 1.1. A function $f: \Omega \to \chi$ is called $|\mu|$-measurable where $\mu:\Sigma \to X$ is a measure of bounded variation, if there exists $N \in \tau(|\mu|)$ $N|\mu|$-negligible, such that there exists a sequence (s_n) of Σ-simple X-valued functions converging to f pointwise in $\Omega \setminus N$. I.e. $s_n \to f$ $|\mu|$-a.e. in Ω.

As μ is of bounded variation, $|\mu|^*$ is bounded in $H(\Sigma)$ and, hence, $|\mu|^*$ is bounded in $\tau(|\mu|)$. Therefore, Ω is $|\mu|$-integrable in the sense of Definition 6, p. 75 of [2]. Consequently, by Theorem 2, p. 99 of [2], a function $f:\Omega \to X$ which is $|\mu|$-measurable in the sense of Definition 4, p. 89 of [2] is $|\mu|$-measurable in the sense of our Definition 1.1. Conversely, as a Σ-simple X-valued function is clearly $|\mu|$-measurable in the sense of [2], by Teorem 1, p. of [2], we obtain that a function $f:\Omega \to X$ which is $|\mu|$-measurable in the sense of Definition 1.1, is $|\mu|$-measurable in the sense of [2].

Thus we have:

PROPOSITION 1.2. Let $\mu: \Sigma \to X$ be a measure of bounded variation $f: \Omega \to X$ is $|\mu|$-measurable in the sense of Definition 1.1 if and

only if it is $|\mu|$-measurable in the sense of [2] .

The theory of integration in §8 of [2] can be simplified to some extent as we have μ defined on the σ-ring Σ.

For a Σ-simple function $f = \sum_{i}^{n} x_i \chi_{E_i}$, $x_i \in X$, $E_i \in \Sigma$, $i=1,2,n$ we define

$$\int_E f d\mu = \sum_{i=1}^{n} \mu(E_i \cap E)x_i, \quad E \in \Sigma \cup \{\Omega\},$$

It is clear that

$$\| \int_E f d\mu \| \leq \int_E \| f \| \, d|\mu|. \qquad (1)$$

DEFINITION 1.3. Let $\mu: \Sigma \to X$ be a measure of bounded variation. If $f: \Omega \to X$ is $|\mu|$-measurable, then we say that f is μ-integrable if there exists a sequence (s_n) of Σ-simple X-valued functions such that

(i) $s_n \to f$ $|\mu|$-a.e in Ω ;

(ii) $\int_\Omega \| s_n - s_m \| d|\mu| \to 0$ as $n,m \to \infty$.

Then by (1), for $E \in \Sigma$, $\left(\int_E s_n \, d\mu \right)_{n=1}^{\infty}$ is a Cauchy sequence in X and is therefore convergent in X. By Propositions 8 and 9, §7 of [2], $\int_E f d\mu = \lim_n \int_E s_n \, d\mu$ is well defined for $E \in \Sigma \cup \{\Omega\}$.

$\mathcal{L}(\mu,X)$ will denote the collection of all X-valued μ-integrable functions. From (i) and (ii) in Definition 1.3, it follows that

$\|f\|$ is $|\mu|$-integrable (in the classical sense) if f is so and that

$$\left\| \int_\Omega f \, d\mu \right\| \leq \int_\Omega \|f\| \, d|\mu| < \infty,$$

as $(\Omega, \Sigma, |\mu|)$ is a finite measure space.

Using the equivalence relation $f \sim g$ if $\{x: f(x) \neq g(x)\}$ is $|\mu|$-negligible, one sees that $L_1(\mu, X) = \mathcal{L}(\mu, X)/_\sim$ is a Banach space under the norm

$$\|f\|_1 = \int_\Omega \|f\| \, d|\mu|.$$

PROPOSITION 1.4. Let $f: \Omega \to X$ be $|\mu|$-measurable. If $\|f\| \in \mathcal{L}(|\mu|, \mathbb{R})$, then $f \in \mathcal{L}(\mu, X)$ and

$$\left\| \int_\Omega f \, d\mu \right\| \leq \int_\Omega \|f\| \, d|\mu| .$$

PROOF. It is obvious that $\|f\|$ is $|\mu|$-measurable. By Proposition 1.2 and by Theorem 2, p. 99 of [2], there exists a sequence (s_n) of Σ-simple X-valued functions such that

(i) $\quad \|s_n(w)\| \leq \|f(w)\|$, $n \in \mathbb{N}$ y $w \in \Omega$

(ii) $\quad s_n \to f \quad |\mu|$-a.e.

Then by Theorem 3, p. 136 of [2], (which applies here), $f \in \mathcal{L}(\mu, X)$ and

$$\left\| \int_\Omega f \, d\mu \right\| \leq \int_\Omega \|f\| \, d|\mu| .$$

§2. PRODUCT MEASURES WITH VALUES IN X

Throughout this section we shall assume that $\mu_i: \Sigma_i \to X$ are measures of bounded variation for $i = 1,2$, where Σ_i are σ-rings of subsets of $\Omega_i \neq \phi$, $i = 1,2$. Then $\sup_{E \in \Sigma_i} |\mu_i|(E) = M_i$ are finite for $i = 1,2$. In this section using arbitrary functions h_E and h^E, $E \in \Sigma_1 \times \Sigma_2$, we prove the existence and uniqueness of the product measures $\mu_1 \times \mu_2$ and $(\mu_1 \times \mu_2)^t$ on $\Sigma_1 \times \Sigma_2$, such that

$$(\mu_1 \times \mu_2)\ (A \times B) = \mu_1(A)\ \mu_2(B)$$

and

$$(\mu_1 \times \mu_2)^t\ (A \times B) = \mu_2(B)\ \mu_1(A)$$

for $A \in \Sigma_1$ and $B \in \Sigma_2$. It is also true that $\mu_1 \times \mu_2$ and $(\mu_1 \times \mu_2)^t$ are of bounded variation in $\Sigma_1 \times \Sigma_2$.

DEFINITION 2.1. Let $E \in \Sigma_1 \times \Sigma_2$. We define the functions $h_E: \Omega_1 \to X$ and $h^E: \Omega_2 \to X$ as follows.

$$h_E(w_1) = \mu_2\{w_2 \in \Omega_2: (w_1,w_2) \in E\},\ w_1 \in \Omega_1$$

and

$$h^E(w_2) = \mu_1\{w_1 \in \Omega_1: (w_1,w_2) \in E\},\ w_2 \in \Omega_2\ .$$

Since $E_{w_1} = \{w_2 \in \Omega_2: (w_1,w_2) \in E\} \in \Sigma_2$ and $E^{w_2} = \{w_1 \in \Omega_1: (w_1,w_2) \in E\} \in \Sigma_1$, as $E \in \Sigma_1 \times \Sigma_2$, the functions h_E and h^E are well defined.

LEMMA 2.2. Let $E \in \Sigma_1 \times \Sigma_2$. Then $h_E: \Omega_1 \to X$ is $|\mu_1|$-measurable

and $h^E: \Omega_2 \to \chi$ is $|\mu_2|$-measurable.

PROOF. We shall prove the result for h_E. In a similar manner the result for h^E can be proved.

Let $E = A \times B$, $A \in \Sigma_1$, $B \in \Sigma_2$. Then, in this case, $h_E = \mu_2(B)\chi_A$ which is clearly $|\mu_1|$-measurable. Consequently, if

$$E = \bigcup_{i=1}^{n} A_i \times B_i, \quad A_i \in \Sigma_1, \quad B_i \in \Sigma_2, \quad (A_i \times B_i) \cap (A_j \times B_j) = \phi, i \neq j,$$

then it is clear that

$$h_E(w_1) = \sum_{i=1}^{n} \mu_2(B_i)\, \chi_{A_i}(w_1)$$

which is a Σ_1-simple χ-valued function and hence is $|\mu_1|$-measurable. Therefore, if R is the ring generated by $\{A \times B: A \in \Sigma_1, B \in \Sigma_2\}$, then h_E is $|\mu_1|$-measurable for each $E \in R$.

Let $M = \{E \in \Sigma_1 \times \Sigma_2: h_E \text{ is } |\mu_1|\text{-measurable}\}$. Then by the foregoing argument, $R \subset M$. Let $\{E_n\}$ be a monotonic sequence in M with $E = \lim_n E_n$. Then $\{(E_n)_{w_1}\}$ is monotonic with $E_{w_1} = \lim_n (E_n)_{w_1}$. Since μ_2 is a vector measure, which is countably additive in Σ_2, it follows that

$$h_E = \mu_2(E_{w_1}) = \lim_n \mu_2((E_n)_{w_1}) = \lim_n h_{E_n}.$$

As $E_n \in M$, h_{E_n} is $|\mu_1|$-measurable for $n \in \mathbb{N}$. Now by Proposition 1.2 and by Theorem 1, p. 94 of [2], $\lim_n h_{E_n} = h_E$ is $|\mu_1|$-measurable. Hence $E \in M$ and consequently, by Theorem B, §8 of [4], we have that $M = S(R) = \Sigma_1 \times \Sigma_2$. That is, h_E is $|\mu_1|$-measurable for each $E \in \Sigma_1 \times \Sigma_2$.

COROLLARY 2.3. Let $\{E_i\}_{i=1}^n \subset \Sigma_1 \times \Sigma_2$. If $\{x_i\}_1^n \subset X$ and if $s: \Omega_1 \to X$ is given by

$$s(w_1) = \sum_{i=1}^{n} h_{E_i}(w_i)x_i$$

then s is $|\mu_1|$-measurable. Similarly $t: \Omega_2 \to X$, given by

$$t(w_2) = \sum_{i=1}^{n} h^{E_i}(w_2)x_i,$$

is $|\mu_2|$-measurable.

PROOF. Let $f_i(w_1) = h_{E_i}(w_1)x_i$, $w_1 \in \Omega_1$. As h_{E_i} is $|\mu_1|$-measurable, by Lemma 2.2, exists a sequence (s_n) of X-valued Σ_1-simple functions which converges to h_{E_i} $|\mu_1|$-a.e. en Ω_1. As $s_n x_i$ is also an X-valued Σ_1-simple function and as $s_n x_i \to f_i$ $|\mu_1|$-a.e in Ω_1, it follows that f_i is $|\mu_1|$-measurable. Now by Proposition 1.2 and by corollary 1, p. 101 of [2], s is $|\mu_1|$-measurable. By a similar argument we also have that t is $|\mu_2|$-measurable.

LEMMA 2.4. For $E \in \Sigma_1 \times \Sigma_2$, $h_E \in \mathcal{L}(\mu_1, X)$ and $h^E \in \mathcal{L}(\mu_2, X)$. If

$$\tau_1(E) = \int_{\Omega_1} h_E \, d\mu_1 \quad \text{and} \quad \tau_2(E) = \int_{\Omega_2} h^E \, d\mu_2 ,$$

then τ_1 and τ_2 are X-valued measures in $\Sigma_1 \times \Sigma_2$. Further,

$$\tau_1(A \times B) = \mu_1(A)\, \mu_2(B) \quad \text{and} \quad \tau_2(A \times B) = \mu_2(B)\, \mu_1(A)$$

for $A \in \Sigma_1$ and $B \in \Sigma_2$.

PROOF. For $E \in \Sigma_1 \times \Sigma_2$, by Lemma 2.2 h_E is $|\mu_1|$-measurable and h^E is $|\mu_2|$-measurable. For $w_1 \in \Omega_1$

$$\| h_E(w_1) \| = \| \mu_2(E_{w_1}) \| \leq |\mu_2| \, (E_{w_1}) \leq M_2$$

and hence $h_E\colon \Omega_1 \to X$ is bounded. As $\| h_E \|$ is $|\mu_1|$-measurable and bounded and $(\Omega_1, \Sigma_1, |\mu_1|)$ is a finite measure space it follows that $\| h_E \|$ is $|\mu_1|$-integrable and consequently, by Proposition 1.4 h_E is μ_1-integrable. Similarly, h^E is μ_2-integrable.

Because of the similarity it is enough to prove that τ_1 is countably additive in $\Sigma_1 \times \Sigma_2$. Let $\{E_i\}_1^\infty \subset \Sigma_1 \times \Sigma_2$, $E = \bigcup_{i=1}^{\infty} E_i$, $E_i \cap E_j = \phi$, $i \neq j$. Then it is obvious that for $w_1 \in \Omega_1$, $\{(E_i)_{w_1}\}$ is a disjoint sequence in Σ_2 and that $E_{w_1} = \bigcup_{i=1}^{\infty} (E_i)_{w_1}$. Further,

for $w_1 \in \Omega_1$

$$h_E(w_1) = \mu_2(E_{w_1}) = \sum_{i=1}^{\infty} \mu_2((E_i)_{w_1}) = \lim_n \sum_{i=1}^{n} \mu_2((E_i)_{w_1})$$

$$= \lim_n h_{\left[\bigcup_{i=1}^{n} E_i\right]}(w_1) . \qquad (1)$$

Also we have that for $w_1 \in \Omega_1$

$$\| h_{\left[\bigcup_{i=1}^{n} E_i\right]}(w_1) \| = \| \mu_2((\bigcup_{i=1}^{n} E_i)_{w_1}) \|$$

$$\leq |\mu_2|((\bigcup_{i=1}^{n} E_i)_{w_1})$$

$$\leq |\mu_2| (E_{w_1}) \leq M_2. \qquad (2)$$

If $\tilde{h}_E: \Omega_1 \to \mathbb{R}$ is given by $\tilde{h}_E(w_1) = |\mu_2| (E_{w_1})$, then from the theory of product measures in the case of positive measures (vide Berberian [1]), we have that $\tilde{h}_E$ is $|\mu_2|$-measurable and is bounded by M_2. Therefore $\tilde{h}_E$ is $|\mu_2|$-integrable as $|\mu_2|$ is a finite measure in Σ_2. We rewrite (2) in terms of $\tilde{h}_E$ as

$$\| h_{\bigcup_1^n E_i}(w_1) \| \leq \tilde{h}_E(w_1), \; w_1 \in \Omega_1. \qquad (2')$$

From (1), (2') and the fact that $\tilde{h}_E \in \mathcal{L}(|\mu_1|, \mathbb{R})$, by theorem 3, p. 136 of [2], we obtain that

$$\int_{\Omega_1} h_E \, d\mu_1 = \lim_n \int_{\Omega_1} h_{\left[\bigcup_1^n E_i\right]} d\mu_1 = \sum_1^{\infty} \int_{\Omega_1} h_{E_i} d\mu_1 .$$

i.e. $$\tau_1(E) = \sum_{1}^{\infty} \tau_1(E_i).$$

Further, for $A \in \Sigma_1$ and $B \in \Sigma_2$, by the definition of the integral of an X-valued simple function, we have that

$$\tau_1(A \times B) = \int_{\Omega_1} h_{A\times B}\, d\mu_1 = \int_{\Omega_1} \mu_2(B)\, \chi_A\, d\mu_1 = \mu_1(A)\mu_2(B)$$

and similarly

$$\tau_2(A \times B) = \int_{\Omega_2} \mu_1(A)\, \chi_B\, d\mu_2 = \mu_2(B)\, \mu_1(A).$$

DEFINITION 2.5. Let $\mathcal{R}$ be the ring generated by the semi-ring $\{A\times B: A \in \Sigma_1, B \in \Sigma_2\}$. We define

$$\mu_1 \times \mu_2: \mathcal{R} \to X \quad \text{by} \quad (\mu_1 \times \mu_2)(E) = \sum_{i=1}^{n} \mu_1(A_i)\, \mu_2(B_i)$$

and

$$(\mu_1 \times \mu_2)^t: \mathcal{R} \to X \text{ by } (\mu_1 \times \mu_2)^t (E) = \sum_{i=1}^{n} \mu_2(B_i)\, \mu_1(A_i).$$

where

$E = \bigcup_{i=1}^{n} (A_i \times B_i)$, $(A_i \times B_i) \cap (A_j \times B_j) = \phi$, $i \neq j$, $A_i \in \Sigma_1$, $B_i \in \Sigma_2$, $i=1,2,\ldots,n$.

THEOREM 2.6. $\mu_1 \times \mu_2$ and $(\mu_1 \times \mu_2)^t$ are well defined in $\mathcal{R}$. There exists a unique extension of $\mu_1 \times \mu_2$ as an X-valued measure, to $\Sigma_1 \times \Sigma_2$. The extension is also denoted by $\mu_1 \times \mu_2$ and is called

the product measure of μ_1 and μ_2 and

$$(\mu_1 \times \mu_2)(E) = \int_{\Omega_1} h_E \, d\mu_1, \quad E \in \Sigma_1 \times \Sigma_2 .$$

A similar result holds for $(\mu_1 \times \mu_2)^t$ and its unique extension, with is also denoted by $(\mu_1 \times \mu_2)^t$, is called the transpose product measure of μ_1 and μ_2. Further, $\mu_1 \times \mu_2$ and $(\mu_1 \times \mu_2)^t$ are of bounded variation in $\Sigma_1 \times \Sigma_2$ and satisfy:

$$|\mu_1 \times \mu_2|(E) \leq (|\mu_1| \times |\mu_2|)(E)$$

and

$$|(\mu_1 \times \mu_2)^t|(E) \leq (|\mu_1| \times |\mu_2|)(E)$$

for $E \in \Sigma_1 \times \Sigma_2$.

PROOF. By Lemma 2.4, we have that $\tau_1(A \times B) = (\mu_1 \times \mu_2)(A \times B)$ and $\tau_2(A \times B) = (\mu_1 \times \mu_2)^t(A \times B)$. If $\mathcal{R}$ is the ring generated by $\{A \times B: A \in \Sigma_1, B \in \Sigma_2\}$, then $\tilde{\tau}_1 = \tau_1 | \mathcal{R}$ and $\tilde{\tau}_2 = \tau_2 | \mathcal{R}$ are countably additive in $\mathcal{R}$ and hence are finitely additive. Hence, if

$$E = \bigcup_{i=1}^{n} (A_i \times B_i), (A_i \times B_i) \cap (A_j \times B_j) = \phi \; i \neq j,$$

$$A_i \in \Sigma_1, B_i \in \Sigma_2, i = 1, 2, \ldots n.$$

then

$$\tilde{\tau}_1(E) = \sum_{i=1}^{n} \tilde{\tau}_1(A_i \times B_i) = \sum_{i=1}^{n} \mu_1(A_i) \, \mu_2(B_i).$$

As $\tilde{\tau}_1$ is well defined on E and is independent of the representation as a finite disjoint union of measurable rectangles, it follows that $(\mu_1 \times \mu_2)(E)$ is well defined for $E \in \mathcal{R}$ and further, as $\mu_1 \times \mu_2 = \tilde{\tau}_1$ en $\mathcal{R}$, $\mu_1 \times \mu_2$ is countably additive on $\mathcal{R}$. Similarly $(\mu_1 \times \mu_2)^t$ is well defined in $\mathcal{R}$ and $(\mu_1 \times \mu_2)^t = \tilde{\tau}_2$ en $\mathcal{R}$.

From Lemma 2.4, it follows that $\mu_1 \times \mu_2$ has a countably additive extension τ_1 and $(\mu_1 \times \mu_2)^t$ has a countably additive extension τ_2 in $\Sigma_1 \times \Sigma_2$. We shall prove the uniqueness of τ_1. Similar arguments will prove the uniqueness of τ_2. If τ_1' is another countably additive X-valued extension of $\mu_1 \times \mu_2$ in $\Sigma_1 \times \Sigma_2$, then for $x^* \in X^*$, $x^* \tau_1(E) = x^* \tau_1'(E)$, $E \in \Sigma_1 \times \Sigma_2$. In fact, for

$$E = \bigcup_{i=1}^{n} A_i \times B_i, \quad (A_i \times B_i) \cap (A_j \times B_j) = \phi, \ i \neq j,$$

$$A_i \in \Sigma_1, \ B_i \in \Sigma_2,$$

we have

$$\| \tau_1(E) \| = \| \sum_{i=1}^{n} \mu_1(A_i) \ \mu_2(B_i) \|$$

$$\leq \sum_{i=1}^{n} \| \mu_1(A_i) \| \ \| \mu_2(B_i) \|$$

$$\leq \sum_{i=1}^{n} |\mu_1|(A_i) \ |\mu_2|(B_i)$$

$$= (|\mu_1| \times |\mu_2|)(E). \quad (1)$$

From the classical theory of product measures of finite positive measures, $|\mu_1| \times |\mu_2| : \Sigma_1 \times \Sigma_2 \to [0,\infty)$ is countably additive and finite (see Theorem 2, p. 126 of [1]). As $\Sigma_1 \times \Sigma_2$ is a σ-ring, it follows that

$$\sup_{E \in \Sigma_1 \times \Sigma_2} (|\mu_1| \times |\mu_2|)(E) = M < \infty. \quad (2)$$

Therefore, from (1) and (2) we obtain that

$$\| \tau_1(E) \| \leq M \quad \text{for all } E \in \mathcal{R}.$$

Consequently, $x^* \tau_1$ is a bounded scalar measure in $\mathcal{R}$ and hence by the uniqueness of the Caratheodory-Hahn extension of bounded scalar measures, we have that $(x^* \tau_1)(E) = x^* \tau_1'(E)$, $E \in \Sigma_1 \times \Sigma_2$. As x^* is arbitrary in X^*, by the Hahn-Banach theorem

$$\tau_1(E) = \tau_1'(E), \quad E \in \Sigma_1 \times \Sigma_2.$$

This completes the proof of existence and uniqueness of the extension of $\mu_1 \times \mu_2$ in $\Sigma_1 \times \Sigma_2$.

From (1), we have that

$$\mathcal{R} \subset \mathcal{M} = \{E \in \Sigma_1 \times \Sigma_2 : |\mu_1 \times \mu_2|(E) \leq (|\mu_1| \times |\mu_2|)(E)\}.$$

As $|\mu_1| \times |\mu_2|$ is a finite measure on $\Sigma_1 \times \Sigma_1$, by a known argument we see that $\mathcal{M}$ is a monotone class and hence by Theorem B, §8 of Halmos [4], $\mathcal{M} = S(\mathcal{R}) = \Sigma_1 \times \Sigma_2$. Therefore,

$$\sup_{E \in \Sigma_1 \times \Sigma_2} |\mu_1 \times \mu_2| \ (E) \leq M$$

and hence $\mu_1 \times \mu_2$ is of bounded variation in $\Sigma_1 \times \Sigma_2$. The corresponding result for $(\mu_1 \times \mu_2)^t$ is proved in a similar manner.

COROLLARY 2.7. If μ_1 and μ_2 commute, in the sense that

$$\mu_1(A)\,\mu_2(B) = \mu_2(A)\,\mu_1(B), \quad A \in \Sigma_1, \ B \in \Sigma_2,$$

then

$$\tau_1 = \tau_2 = \mu_1 \times \mu_2 = (\mu_1 \times \mu_2)^t \text{ in } \Sigma_1 \times \Sigma_2.$$

PROOF. The result follows from Lemma 2.4 and the uniqueness part of Theorem 2.6.

§3. A GENERALIZED FUBINI'S THEOREM

Let Ω_i, Σ_i, μ_i, M_i, $i=1,2$ be as in §2. Let $\mu_1 \times \mu_2$ and $(\mu_1 \times \mu_2)^t$ be the product and transpose product measures respectively. In this section we obtain a generalized Fubini's theorem, giving the relation between the integral with respect to the product (transpose product) measure and a suitable iterated integral, when the function $f: \Omega_1 \times \Omega_2 \to X$ is $|\mu_1| \times |\mu_2|$-measurable and $\|f\|$ is $|\mu_1| \times |\mu_2|$-integrable. Further, when μ_1 and μ_2 commute and the range of f is bounded in X, this result reduces to the theorem of Fubini for such integrals.

Ω will denote $\Omega_1 \times \Omega_2$ in the sequel.

LEMMA 3.1. Let $f: \Omega \to X$ be such that f is $|\mu_1| \times |\mu_2|$-measurable and $\|f\| \in \mathcal{L}(|\mu_1| \times |\mu_2|, \mathbb{R})$. Then f is $(\mu_1 \times \mu_2)$ and $(\mu_1 \times \mu_2)^t$-integrable.

PROOF. Let $f_{w_1}(w_2) = f(w_1, w_2) = f^{w_2}(w_1)$. As f is $|\mu_1| \times |\mu_2|$-measurable, there exist a sequence (s_n) of X-valued $\Sigma_1 \times \Sigma_2$-simple functions and a set N, $(|\mu_1| \times |\mu_2|)$ - negligible such that $s_n \to f$ in $\Omega \setminus N$. As $|\mu_1 \times \mu_2|(E) \leq |\mu_1| \times |\mu_2|(E)$ for $E \in \Sigma_1 \times \Sigma_2$ by Theorem 2.6, it is clear that N is $|\mu_1 \times \mu_2|$-negligible (vide. Proposition 11, p. 15 of [2]) and hence $s_n \to f$ $|\mu_1 \times \mu_2|$-a.e. Therefore, by Definition 1.1, f is $|\mu_1 \times \mu_2|$-measurable. Consequently, $\|f\|$ is $|\mu_1 \times \mu_2|$-measurable. Further,

$$\int_\Omega \|f\| \, d|\mu_1 \times \mu_2| \leq \int_\Omega \|f\| \, d(|\mu_1| \times |\mu_2|) < \infty$$

and hence by Proposition 1.4, f is $(\mu_1 \times \mu_2)$-integrable.

Since the inequality $|(\mu_1 \times \mu_2)^t|(E) \leq |\mu_1| \times |\mu_2|(E)$, $E \in \Sigma_1 \times \Sigma_2$ holds by Theorem 2.6, the above argument can be modifield to prove that f is $(\mu_1 \times \mu_2)^t$-integrable.

In the sequel we shall assume that $f: \Omega \to X$ is $|\mu_1| \times |\mu_2|$-measurable and that $\|f\|$ is $|\mu_1| \times |\mu_2|$-integrable.

LEMMA 3.2. Let N be $|\mu_1| \times |\mu_2|$-negligible. Then there exist sets A and B such that $\Omega \setminus N = A \cup B$, where $A \in \Sigma_1 \times \Sigma_2$ and B is $|\mu_1| \times |\mu_2|$-negligible. If the function $g: \Omega \to X$ satisfies the relation $g(w_1, w_2) = f(w_1, w_2)\, \chi_A$, then there exist $C \subset \Omega_1$, $D \subset \Omega_2$, $C|\mu_1|$-negligible, $D|\mu_2|$-negligible, g is $|\mu_1| \times |\mu_2|$-measurable, $g_{w_1} = f_{w_1}$ $|\mu_2|$-a.e. and $g^{w_2} = f^{w_2}$ $|\mu_1|$-a.e., for $w_1 \in \Omega_1 \setminus C$ and for $w_2 = \Omega_2 \setminus D$.

PROOF. Since $\sup_{E \in \Sigma_1 \times \Sigma_2} (|\mu_1| \times |\mu_2|)(E) < \infty$, $\Omega \setminus N$ is a $|\mu_1| \times |\mu_2|$-integrable set in the sense of Definition 6, p. 75 of [2]. Hence by Proposition 12, p. 75 of [2], there exist sets $A \in \Sigma_1 \times \Sigma_2$ and B, $|\mu_1| \times |\mu_2|$-negligible, such that $\Omega \setminus N = A \cup B$. Therefore, $g = f\, \chi_A$ is $|\mu_1| \times |\mu_2|$-measurable by Corollary 1, p. 101 of [2].

Let $h = f - g$. Then $h(w_1, w_2) = 0$ for $(w_1, w_2) \in A$ and hence $h = 0$ $|\mu_1| \times |\mu_2|$-a.e. Now, by the classical Fubini's theorem, for $E \in \tau(|\mu_1|)$, $F \in \tau(|\mu_2|)$

$$0 = \int_{E \times F} \| h \|\, d(|\mu_1| \times |\mu_2|) = \int_E \left(\int_F \| h_{w_1} \|\, d|\mu_2| \right) d|\mu_1|$$

$$= \int_F \left(\int_E \| h^{w_2} \|\, d|\mu_1| \right) d|\mu_2|$$

where we consider the restrictions of $|\mu_1|$ and $|\mu_2|$ in

$$\tau_E(|\mu_1|) = \{G \in \tau(|\mu_1|): G \subset E\}$$

and

$$\tau_F(|\mu_2|) = \{G \in \tau(|\mu_2|): H \subset F\}$$

respectively. Fixing F and varying E we obtain that there exists $C \subset \Omega_1$, C -$|\mu_1|$-negligible such that

$$\int_F \| h_{w_1} \| \, d|\mu_2| = 0 \qquad \text{for } w_1 \in \Omega_1 \setminus C$$

and now varying F in $\tau(|\mu_2|)$, we deduce that $h_{w_1} = 0$ $|\mu_2|$-a.e. Similarly, we have that $h^{w_2} = 0$ $|\mu_1|$-a.e., for $w_2 \in \Omega \setminus D$, $D|\mu_2|$--negligible. Hence $f_{w_1} = g_{w_1}$ $|\mu_2|$-a.e. and $f^{w_2} = g^{w_2}$ $|\mu_1|$-a.e., for such w_1 and w_2.

LEMMA 3.3. There exist sets $N_i \subset \Omega_i$, N_i $|\mu_i|$-negligible, $i=1,2$ such that

(i) for $w_1 \in \Omega_1 \setminus N_1$, $\int_{\Omega_2} f_{w_1} \, d\mu_2$ exists and

(ii) for $w_2 \in \Omega \setminus N_2$, $\int_{\Omega_1} f^{w_2} \, d\mu_1$ exists.

PROOF. As f is $|\mu_1| \times |\mu_2|$-measurable, there exist N $|\mu_1| \times |\mu_2|$-negligible and a sequence (s_n) of $\Sigma_1 \times \Sigma_2$-simple X-valued functions such that $s_n \to f$ in $\Omega \setminus N$. By Lemma 3.3, there exist sets A and B, with $A \in \Sigma_1 \times \Sigma_2$ and B $|\mu_1| \times |\mu_2|$-negligible, such that $\Omega \setminus N = A \cup B$. Let $g = f \chi_A$. Then $s_n \chi_A \to g$ in Ω and

$t_n = s_n \chi_A$ is $\Sigma_1 \times \Sigma_2$-simple. Consequently,

$$g_{w_1}(w_2) = \lim_n (t_n)_{w_1}(w_2), \quad w_2 \in \Omega_2$$

and hence g_{w_1} is $|\mu_2|$-measurable. Similarly, g^{w_2} is $|\mu_1|$-measurable.

Now from Lemma 3.2, it follows that $g_{w_1} = f_{w_1}$ $|\mu_2|$-a.e. for $w_1 |\mu_1|$-a.e., and $g^{w_2} = f^{w_2}$, $|\mu_1|$-a.e., for $w_2 |\mu_2|$-a.e. Consequently, by Proposition 1.2. and by Proposition 9, p. 91 of [2], we have that f_{w_1} is $|\mu_2|$-measurable and f^{w_2} is $|\mu_1|$-measurable, for $w_1 |\mu_1|$-a.e. and $w_2 |\mu_2|$-a.e.

As $\| f \| \in \mathcal{L}(|\mu_1| \times |\mu_2|, \mathbb{R})$, applying the classical Fubini's theorem to $\| f \|$, we obtain that

$$\int_{\Omega_1} \| f^{w_2} \| \, d|\mu_1| < \infty$$

for $w_2 \in \Omega_2 \setminus \tilde{N}_2$, $\tilde{N}_2$ $|\mu_2|$-negligible and

$$\int_{\Omega_2} \| f_{w_1} \| \, d|\mu_2| < \infty$$

for $w_1 \in \Omega \setminus \tilde{N}_1$, $\tilde{N}_1$ $|\mu_1|$-negligible. Consequently, we have, by Proposition 1.4, that $\int_{\Omega_2} f_{w_1} \, d\mu_2$ exists for $w_1 \in \Omega_1 \setminus N_1$ and $\int_{\Omega_1} f^{w_2} \, d\mu_1$ exists for $w_2 \in \Omega_2 \setminus N_2$, where $N_1 = \tilde{N}_1 \cup C$ y $N_2 = \tilde{N}_2 \cup D$, C and D being those given in Lemma 3.2.

LEMMA 3.4. With N_1, N_2 as in Lemma 3.3, let

$$F(w_1) = \int_{\Omega_2} f_{w_1} \, d\mu_2, \quad w_1 \in \Omega_1 \setminus N_1$$

$$= 0, \quad w_1 \in N_1;$$

and

$$G(w_2) = \int_{\Omega_1} f^{w_2} \, d\mu_1, \quad w_2 \in \Omega_2 \setminus N_2$$

$$= 0, \quad w_2 \in N_2.$$

Then F is $|\mu_1|$-measurable and G is $|\mu_2|$-measurable.

PROOF. As f is $|\mu_1| \times |\mu_2|$-integrable by Lemma 3.1, there exists a sequence (s_n) of X-valued $\Sigma_1 \times \Sigma_2$-simple fuctions such that

$$\int_{\Omega} \| s_n - f \| \, d(|\mu_1| \times |\mu_2|) < \frac{1}{n}, \quad n \in \mathbb{N} \qquad (1)$$

and that $s_n \to f$ $|\mu_1| \times |\mu_2|$-a.e. By the classical Fubini's theorem, there exists $\tilde{N}_n$, $|\mu_1|$-negligible, such that

$$F_n(w_1) = \int_{\Omega_2} \| (s_n)_{w_1} - f_{w_1} \| \, d|\mu_2| < \infty, \quad w_1 \in \Omega_1 \setminus \tilde{N}_n,$$

$F_n \in \mathcal{L}(|\mu_1|, \mathbb{R})$ and

$$\int_{\Omega_1} F_n \, d|\mu_1| = \int_{\Omega} \| s_n - f \| \, d(|\mu_1| \times |\mu_2|) < \frac{1}{n}$$

by (1). Therefore,

$$\lim_{n\to\infty} \int_{\Omega_1} F_n \, d|\mu_1| = 0 .$$

That is, F_n tends to 0 in $|\mu_1|$-mean and hence by Proposition 1.4 p. 130 of [2], there exists a subsequence (F_{n_k}) of (F_n) such that $F_{n_k} \to 0$ $|\mu_1|$-a.e. Therefore, exists a set N_o, $|\mu_1|$-negligible, such that $F_{n_k}(w_1) \to 0$ for all $w_1 \in \Omega_1 \setminus N_o$. Let

$$\tilde{N} = N_1 \cup \bigcup_{k=1}^{\infty} \tilde{N}_k \cup N_o .$$

Then $\tilde{N}$ is $|\mu_1|$-negligible. For $w_1 \in \Omega_1 \setminus \tilde{N}$,

$$\| F(w_1) - \int_{\Omega_2} (s_{n_k})_{w_1} \, d\mu_2 \| = \| \int_{\Omega_2} \{ f_{w_1} - (s_{n_k})_{w_1} \} \, d\mu_2 \|$$

$$\leq \int_{\Omega_2} \| f_{w_1} - (s_{n_k})_{w_1} \| \, d|\mu_2|$$

$$= F_{n_k}(w_1) \to 0$$

as $k \to \infty$.

In other words, we have that

$$F(w_1) = \lim_{k} \int_{\Omega_2} (s_{n_k})_{w_1} \, d\mu_2, \quad |\mu_1|\text{-a.e.} \quad (2)$$

We observe that if

$$s = \sum_{i=1}^{n} x_i \chi_{E_i}, \quad x_i \in X, \quad E_i \in \Sigma_1 \times \Sigma_2, \quad i=1,2,\ldots,n,$$

so that s es $\Sigma_1 \times \Sigma_2$- simple with values in X, then

$$\int_{\Omega_2} s_{w_1} \, d\mu_2 = \sum_{i=1}^{n} \mu_2((E_i)_{w_1})x_i = \sum_{i=1}^{n} h_{E_i}(w_1)x_i.$$

Now, by Corollary 2.3, we obtain that $\int_{\Omega_2} s_{w_1} \, d\mu_2$ is $|\mu_1|$-measurable. Now by (2) and by Theorem 1, p. 94 of [2] we have that F is $|\mu_1|$-measurable.

The proof for G is similar and hence omitted.

THEOREM 3.5. (A Generalized Fubini's Theorem). Let $f: \Omega \to X$ be $|\mu_1| \times |\mu_2|$-measurable with $\|f\|$, $|\mu_1| \times |\mu_2|$-integrable. Then there exist $N_i \in \Sigma_i$ with N_i $|\mu_i|$-negligible, i=1,2, such that for $w_1 \in \Omega_1 \setminus N_1$, there exists the integral $\int_{\Omega_2} f_{w_1} \, d\mu_2$ and for $w_2 \in \Omega_2 \setminus N_2$ there exists the integral $\int_{\Omega_1} f^{w_2} \, d\mu_1$.

If $F: \Omega_1 \to X$ and $G: \Omega_2 \to Y$ are defined in such a way that $F(w_1) = \int_{\Omega_2} f_{w_1} \, d\mu_2$, $w_1 \in \Omega_1 \setminus N_1$ and $G(w_2) = \int_{\Omega_1} f^{w_2} \, d\mu_1$, $w_2 \in \Omega_2 \setminus N_2$,

then F is μ_1-integrable and G is μ_2-integrable. Further, we have that

$$\int_\Omega f \, d(\mu_1 \times \mu_2) = \int_{\Omega_1} F \, d\mu_1 = \int_{\Omega_1} \int_{\Omega_2} f_{w_1} \, d\mu_2 \, d\mu_1$$

and

$$\int_\Omega f \, d(\mu_1 \times \mu_2)^t = \int_{\Omega_2} G \, d\mu_2 = \int_{\Omega_2} \int_{\Omega_1} f^{w_2} \, d\mu_1 . d\mu_2 .$$

PROOF. By Lemma 3.4 and by Proposition 9, p. 91 of [2], F is $|\mu_1|$-measurable and G is $|\mu_2|$-measurable, and the existence of N_i with specified properties is guaranteed by Lemma 3.3.

From (1) in the proof of Lemma 3.4, there exists a sequence (s_n) of X-valued $\Sigma_1 \times \Sigma_2$-simple functions such that

$$\int_\Omega \| s_n - f \| \, d(|\mu_1| \times |\mu_2|) < \frac{1}{n} \quad (1)$$

for $n \in \mathbb{N}$. Let

$$s_n = \sum_{i=1}^{n} x_i \chi_{E_i}, \quad E_i \in \Sigma_1 \times \Sigma_2, \; E_i \cap E_j = \phi, \; i \neq j.$$

If

$$s(w_1) = \int_{\Omega_2} (s_n)_{w_1} \, d\mu_2 ,$$

then

$$s(w_1) = \sum_{i=1}^{n} h_{E_i}(w_i)x_i.$$

Then by Corollary 2.3, s is $|\mu_1|$-measurable and

$$\|s(w_1)\| \leq \sum_{i=1}^{n} \|h_{E_i}(w_1)\| \|x_i\|$$

$$\leq \left(\max_{1\leq i\leq n} \|x_i\|\right) \left(\sum_{i=1}^{n} \|\mu_2((E_i)_{w_1})\|\right)$$

$$\leq K\left(\sum_{i=1}^{n} |\mu_2|((B_i)_{w_1})\right)$$

$$= K|\mu_2|\left(\left(\bigcup_{i=1}^{n} E_i\right)_{w_1}\right)$$

$$\leq K M_2 < \infty$$

where $K = \max_{1\leq i\leq n} \|x_i\|$. Hence $\|s\|$ is a bounded $|\mu_1|$-measurable function and so $\|s\|$ is $|\mu_1|$-integrable. Consequently, by Proposition 1.4, s is μ_1-integrable.

Now we shall prove that F is μ_1-integrable. It is clear that $\|F-s\|$ is $|\mu_1|$-measurable and

$$\int_{\Omega_1} \|F-s\| \, d|\mu_1|$$

$$= \int_{\Omega_1} \| \int_{\Omega_2} (f_{w_1} - (s_n)_{w_1}) \, d\mu_2 \| \, d|\mu_1|$$

$$\leq \int_{\Omega_1} \int_{\Omega_2} \|f_{w_1} - (s_n)_{w_1}\| \, d|\mu_2| \, d|\mu_1|$$

$$= \int_{\Omega} \| f - s_n \| \, d(|\mu_1| \times |\mu_2|)$$

$$< \frac{1}{n} \qquad (2)$$

by the classical Fubini's theorem and by (1).

Let $t_n(w_1) = \int_{\Omega_2} (s_n)_{w_1} \, d\mu_2$. Then, by the foregoing, t_n is μ_1-integrable and consequently, from (2) it follows that

$$\int_{\Omega_1} \| F \| \, d|\mu_1| \leq \int_{\Omega_2} \| F - t_n \| \, d|\mu_1| + \int_{\Omega_1} \| t_n \| \, d|\mu_1| < \infty.$$

Therefore, $\| F \|$ is $|\mu_1|$-integrable and hence by Proposition 1.4 F is μ_1-integrable.

By a similar argument it follows that G is μ_2-integrable.

AFFIRMATION 1. If

$$s = \sum_{i=1}^{n} x_i \, \chi_{E_i}, \quad E_i \cap E_j = \phi, \; i \neq j, \; E_i \in \Sigma_1 \times \Sigma_2, \; x_i \in X$$

then

$$\int_{\Omega} s\, d(\mu_1 \times \mu_2) = \int_{\Omega_1} \int_{\Omega_2} s_{w_1}\, d\mu_2\, d\mu_1$$

In fact,

$$\int_{\Omega} s\, d(\mu_1 \times \mu_2) = \sum_{i=1}^{n} (\mu_1 \times \mu_2)(E_i) x_i = \sum_{i=1}^{n} \tau_1(E_i) x_i \ , \quad (3)$$

by Theorem 2.6.

We observe that if g is $\Omega_1 \to \mathbf{X}$ is μ_1-integrable and $x \in X$ is a fixed vector in X, then gx is μ_1-integrable and

$$\left(\int_{\Omega_1} g\, d\mu_1 \right) x = \int_{\Omega_1} gx\, d\mu_1 .$$

Using this observation in (3), we have

$$\int_{\Omega} s\, d(\mu_1 \times \mu_2) = \sum_{i=1}^{n} \left(\int_{\Omega_1} h_{E_i}\, d\mu_1 \right) x_i$$

$$= \sum_{i=1}^{n} \int_{\Omega_1} h_{E_i} x_i\, d\mu_1$$

$$= \int_{\Omega_1} \left(\sum_{i=1}^{n} h_{E_i} x_i \right) d\mu_1$$

$$= \int_{\Omega_1} \left[\sum_{i=1}^{n} \mu_2((E_i)_{w_1}) x_i \right] d\mu_1$$

$$= \int_{\Omega_1} \Big(\int_{\Omega_2} \Big(\sum_1^n x_i \, \chi_{(E_i)_{w_1}} \Big) d\mu_2 \Big) d\mu_1$$

$$= \int_{\Omega_1} \int_{\Omega_2} s_{w_1} \, d\mu_2 \, d\mu_1 \, .$$

AFFIRMATION 2. Let s be as in affirmation 1. Then

$$\int_{\Omega} s \, d(\mu_1 \times \mu_2)^t = \int_{\Omega_2} \int_{\Omega_1} s^{w_2} \, d\mu_1 \, d\mu_2 \, .$$

In fact, using τ_2 of Lemma 2.4 in place of τ_1 in the proof of Affirmation 1, the present affirmation can be proved.

From (2), recalling that $s = t_n$, we have

$$\int_{\Omega_1} F \, d\mu_1 = \lim_n \int_{\Omega_1} t_n \, d\mu_1$$

$$= \lim_n \int_{\Omega_1} \int_{\Omega_2} (s_n)_{w_1} \, d\mu_2 \, d\mu_1 \, . \quad (4)$$

Using Affirmation 1 in (4), we obtain

$$\int_{\Omega_1} F \, d\mu_1 = \lim_n \int_{\Omega} s_n \, d(\mu_1 \times \mu_2)$$

$$= \int_{\Omega} f \, d(\mu_1 \times \mu_2)$$

as

$$\left\| \int_{\Omega} f \, d(\mu_1 \times \mu_2) - \int_{\Omega} s_n \, d(\mu_1 \times \mu_2) \right\| \leq \int_{\Omega} \| f - s_n \| \, d(|\mu_1| \times |\mu_2|) <$$

$$< \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly, using Affirmation 2 we have

$$\int_{\Omega_2} G \, d\mu_2 = \int_{\Omega} f \, d(\mu_1 \times \mu_2)^t .$$

This completes the proof of the theorem.

COROLLARY 3.6. (Fubini's Theorem) In addition, let μ_1 and μ_2 be commuting measures. If $f: \Omega \to X$ is a bounded $|\mu_1| \times |\mu_2|$-measurable function, then f is $\mu_1 \times \mu_2$-integrable and

$$\int_{\Omega} f \, d(\mu_1 \times \mu_2) = \int_{\Omega_1} \int_{\Omega_2} f_{w_1} \, d\mu_2 \, d\mu_1 = \int_{\Omega_2} \int_{\Omega_1} f^{w_2} \, d\mu_1 \, d\mu_2 .$$

PROOF. $\| f \|$ is $|\mu_1| \times |\mu_2|$-measurable and bounded. Hence, as $|\mu_1| \times |\mu_2|$ is a finite measure in $\Sigma_1 \times \Sigma_2$, $\| f \|$ is $|\mu_1| \times |\mu_2|$-integrable and hence by Lemma 3.1, f is $\mu_1 \times \mu_2$-integrable. Now by Theorem 3.5 and Corollary 2.7, the conclusion is obtained.

It is known that for two complex measures μ_1 and μ_2 on σ-algebras Σ_1 and Σ_2 respectively, $|\mu_1 \times \mu_2| = |\mu_1| \times |\mu_2|$ (Vide Lemma III. 11.11 of [3]). But this is not generally true

for X-valued measures as we see in the following counter example.

COUNTER EXAMPLE 3.7. Let X be a commutative Banach algebra with $x_1, x_2 \in X$ such that $\| x_1 x_2 \| < \| x_1 \| \| x_2 \|$. Let Σ_1 and Σ_2 be σ-algebras of subsets of Ω_1 and Ω_2 respectively, with $\Omega_1 \neq \phi$ and $\Omega_2 \neq \phi$. Let $\nu_i : \Sigma_i \to [0,\infty)$ be non-negative non trivial measures, $i = 1,2$, then $\mu_i = x_i \nu_i$, $i=1,2$, are X-valued measures on Σ_i with

$$|\mu_i| = \| x_i \| \nu_i \ , \ i=1,2$$

$$|\mu_1 x \mu_2| = \| x_1 x_2 \| (\nu_1 \times \nu_2)$$

$$|\mu_1| \times |\mu_2| = \| x_1 \| \ \| x_2 \| (\nu_1 \times \nu_2) \ .$$

Consequently, $(\mu_1 \times \mu_2)(\Omega_1 \times \Omega_2) = \| x_1 x_2 \| (\nu_1 \times \nu_2)(\Omega_1 \times \Omega_2)$

$= \| x_1 x_2 \| \nu_1(\Omega_1) \nu_2(\Omega_2) < \| x_1 \| \ \| x_2 \| \nu_1(\Omega_1) \nu_2(\Omega_2) =$

$= (|\mu_1| \times |\mu_2|)(\Omega_1 \times \Omega_2)$.

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