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Abstract. Hereditarily locally compact spaces are characterized as those locally compact spaces, which are simple extensions of discrete spaces.

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Introduction. If a simple extension of a discrete space is locally compact, then it is hereditarily so. Surprisingly the converse also holds, i.e. every hereditarily locally compact space is a simple extension of a discrete space. The aim of this note is to prove this fact.

Preliminaries. All spaces in question are supposed to be Hausdorff. A dense embedding $e:(X,T) \to (Y,S)$ is called a simple extension of (X,T), provided $e[X]$ is open in (Y,S) and the subspace of (Y,S), determined by the set $Y \setminus e[X]$, is discrete (see B. Banaschewski [1]). A simple extension $e:(X,T) \to (Y,S)$ of (X,T) is called a simple local compactification of (X,T), provided (Y,S) is locally compact. A simple local compactification $e: (X,T) \to (Y,S)$ is called maximal, provided there does not exist any proper local compactification $c:(Y,S) \to (Z,R)$ such that $c.e: (X,T) \to (Z,R)$ is a simple extension of (X,T). A space (Y,S) is called a (maximal) simple local compactification of a space (X,T), provided there exists a map $e:X \to Y$ such that $e:(X,T) \to (Y,S)$ is a (maximal) simple local compactification.

For any set A and any set B of infinite subsets of A such that any two members of B have finite intersection, we will construct a simple local compactification $e_{(A,B)}: (A,PA) \to (X_{(A,B)}, T_{(A,B)})$ as follows: (1) $PA = \{C \,/\, C \subset A\}$ is the discrete topology on A. (2) $X_{(A,B)}$ is the disjoint union of A and B. (3) $e: A \to X_{(A,B)}$ is the natural embedding, defined by $e(a) = a$ for each $a \in A$. (4) $T_{(A,B)}$ is the set of those subsets D of $X_{(A,B)}$, satisfying the condition that $B \in D \cap B$ implies that $B \setminus D$ is finite. [In B.Banaschewski's suggestive terminology: $e_{(A,B)}: (A,PA) \to (X_{(A,B)}, T_{(A,B)})$ is the simple extension of the discrete space (A,PA), determined by the family $[F_B \,/\, B \in B)$ of trace-filters $F_B = \{C \subset B \mid B \quad C \text{ finite}\}$].

Results

Theorem 1. For any space (Y,S) the following conditions are equivalent.

(1) (Y,S) is hereditarily locally compact,

(2) (Y,S) is a simple local compactificatión of a discrete space,

(3) (Y,S) is homeomorphic to a space $(X_{(A,B)}, T_{(A,B)})$ for suitable A and B.

Proof: (1) => (2). Let X be the set of all isolated points of (Y,S), let T be the discrete topology on X, and let $e: X \to Y$ be the natural embedding, defined by $e(x) = x$ for each $x \in X$. We will show that the embedding $e: (X,T) \to (Y,S)$ is a simple local compactificatión. First, assume $e[X] = X$ were not dense in (Y,S). Then there would exist a non-empty, open subset A of $Y \setminus X$ with compact closure. Hence there would exist a sequence of pairwise disjoint open subsets A_n of A, a sequence of elements $a_n \in A_n$, and an adherence point y of $\{a_n | n \in \mathbb{N}\}$. Consequently the subspace of (Y,S), determined by the set $\{y\} \cup \bigcup\{A_n \setminus \{a_n\} \mid n \in \mathbb{N}\}$, would not be locally compact at y, contradicting condition (1). Hence $e:(X,T) \to (Y,S)$ is an open, dense embedding. It remains to show that the subspace (Z,R) of (Y,S), determined by the set $Z = Y \setminus X$, is discrete. To see this, let z be an element of Z. Since the subspace of (Y,S), determined by the set $X \cup \{z\}$, is locally compact there exists a neighbourhood U of z in (Y,S) such that $U \cap (X \cup \{z\})$ is compact. This implies $Z \cap \operatorname{int}_{(Y,S)} U = \{z\}$, since otherwise $U \cap (X \cup \{z\})$ would not be closed in (Y,S) and hence could not be compact. Therefore z is isolated in (Z,R), hence (Z,R) is discrete.

(2) => (3) Let (X,T) be a discrete space and $e:(X,T) \to (Y,S)$ be a simple local compactification. For each $y \in Y \setminus e[X]$, the set $e[X] \cup \{y\}$ is a neighbourhood of y in (Y,S). Hence there exists a compact neighbourhood K_y of y in (Y,S) with $K_y \subset e[X] \cup \{y\}$. Since $e[X]$ is dense in (Y,S) and (Y,S) is a Hausdorff space, each set K_y is infinite. Since the subspace of (Y,S), determined by K_y, is compact and $e[X]$ consists of isolated points only, every neighbourhood of y meets every infinite subset of K_y. By Hausdorffness of (Y,S) this implies that $K_y \cap K_z$ is finite for any two different elements y and z of $Y \setminus e[X]$. With $A = X$ and $B = \{K_y \ \{y\} \mid y \in Y \setminus e[X]\}$, the

extensions $e_{(A,B)}:(X,T) \to (X_{(A,B)}, T_{(A,B)})$ and $e(X,T) \to (Y,S)$ are obviously equivalent. In particular, the spaces $(X_{(A,B)}, T_{(A,B)})$ and (Y,S) are homeomorphic.

(3) => (1) Straightforeward.

Corollary. Every hereditarily locally compact space is scattered, sequential, and an extension of a discrete space, which is simultaneously simple and strict.

Theorem 2. For any space (Y,S) the following conditions are equivalent:

(1) (Y,S) is a maximal simple local compactification of a discrete space,

(2) (Y,S) is pseudocompact and hereditarily locally compact,

(3) (Y,S) is homeomorphic to a space $(X_{(A,B)}, T_{(A,B)})$, where A is a set and B is a set of infinite subsets of A, which is maximal with respect to the property that any two of its members have finite intersection,

(4) (Y,S) is regular, a simple extension of a discrete space, and every closed set of isolated points in (Y,S) is finite.

Proof: (1) => (2). Let (X,T) be a discrete space and let $e:(X,T) \to (Y,S)$ be a maximal simple local compactification of (X,T). According to theorem 1, the space (Y,S) is hereditarily locally compact. If it were not pseudocompact, there would exist a sequence (y_n) in $e[X]$ and a continuous map f from (Y,S) into the reals with $\lim_{n\to\infty} f(y_n) = \infty$. This would - in contradiction to (1) - allow the construction of a proper local compactification $c:(Y,S) \rightarrow (Z,R)$ of (Y,S) such that $c \circ e:(X,T) \rightarrow (Z,R)$ were simple. As Z one could chose the disjoint union of Y with a singleton set $\{z_o\}$, as $c: Y \rightarrow Z$ the natural embedding, as topology R the set of all subsets R of Z satisfying the following 2 conditions:

(a) $R \cap Y \in S$

(b) if $z_o \in R$ then $\{y_n\} n \in \mathbb{N}\} \setminus R$ is finite.

(2) => (3), According to theorem 1, (Y,S) is homeomorphic to a space $(X_{(A,B)}, T_{(A,B)})$ for suitable A and B. If B would not be maximal, there would exist an infinite subset C of A, meeting each $B \in B$ in at most finitely many points. Hence C would determine an infinite, clopen, discrete subspace of $(X_{(A,B)}, T_{(A,B)})$, contradicting the pseudocompactness of the latter.

(3) => (4). Straightforeward.

(4) => (1). Let (X,T) be a discrete space and let $e: (X,T) \rightarrow$ $\rightarrow (Y,S)$ be a simple extension of (X,T). For any $y \in Y$, the set $e[X] \cup \{y\}$ is a neighbourhood of y in (Y,S). Hence there exists a closed neighbhourhood U of y with $U \subset e[X] \cup \{y\}$. For any neighbourhood V of y, the set $U \setminus V$ is a closed set of isolated points in (Y,S), and hence finite. Consequently U is a compact neighbhourhood of y. Thus $e \cdot (X,T) \rightarrow (Y,S)$ is a simple local compactification of (X,T). To show maximality, let $c:(Y,S) \rightarrow (Z,R)$ be a local compactification of (Y,S) such that $c.e.(X,T) \rightarrow (Z,R)$ is a simple extension of (X,T). Then $c \cdot (Y,S) \rightarrow (Z,R)$

must be improper, since otherwise there would exist an element $z \in Z \setminus c[Y]$ and a compact neighbourhood K of z in (Z,R) with $K \subset c.e[X] \cup \{z\}$. Consequently $c^{-1}[K]$ would be an infinite, closed subset of isolated points in (Y,S), contradicting condition (4).

Reference

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