V. KANNAN AND M. RAJAGOPALAN

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V. KANNAN AND M. RAJAGOPALAN

DEPARTAMENTO DE MATEMATICA

FACULTAD DE CIENCIAS

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V. KANNAN AND M. RAJAGOPALAN*

ABSTRACT: We first obtain some neat characterizations of hereditarily locally compact separable spaces. The first section includes also some characterizations of minimal one among them. The second section describes intrinsically the one-point-compactifications of such spaces. It is also proved that a compact Hausdorff sequential space of type (2,1) fails to be Frechet if and only if it contains one such space as a subspace. Thus a good class of test spaces for Frechet property is obtained here in answer to a problem of Arhangelskii and Franklin. In constrast no this we like to mention that it was proved in [R1] that S_2 cannot be a test space for sequential spaces of order 2.

Key words:- Sequential space, type (2,1), test space, Ψ^*,
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Throughout, N denotes the set of all natural numbers; ω_1 is the first uncountable ordinal number; 'p.a.d.' is the abbreviation for 'pairwise almost disjoint'. Two sets A and B are said to be almost disjoint if their intersection is finite; they are said to be almost equal if their summetric difference is finite; if $A \setminus B$ is finite, we say that A almost contains B. We use the word 'clopen' in the sense of 'both closed and open'. 'h.l.c.s.' is our abbreviation for 'hereditarily-locally compact and separable'. The h.l.c.s. spaces may contain monseparable subspaces.

Let $\underline{F}$ be a family of infinite, p.a.d., subsets of N. Let $\psi(\underline{F})$ denote the disjoint set union $N \cup \underline{F}$, topologized as bolow. If F is an element of $\underline{F}$, it is on one hand an element of $\psi(\underline{F})$, and on the other hand a subset of N and hence of $\psi(\underline{F})$, Since this may create some confusion in our later discussions, we shall denote by F^*, the element F of $\underline{F}$, when viewed as an element of $\psi(\underline{F})$. Thus we have

$$\psi(\underline{F}) = N \cup \{F^* : F \in \underline{F}\}.$$

Each element of N is declared to be isolated in $\psi(\underline{F})$. If $F \in \underline{F}$, a set containing F^* is defined to be a neighbourhood of F^* if and only if it almost contains F.

An easy application of Zorn's lemma shows that every such $\underline{F}$ is contained in a maximal family of infinite p.a.d. subsets of N. The spaces $\psi(\underline{F})$ for such maximal families $\underline{F}$, play a special role in this paper.

Remarks: These spaces were first introduced by J.R. Isbell and considered in [G-J, p. 79]; they appear in [F] to provide an example of a compact Hausdorff sequential non-Frechet space; [R1] contains some non-embeddability theorems concerning them; [K1] and [R2] have used them to construct compact Hausdorff spaces of higher sequential orders; [K2] speaks of them as the spaces that let to guess the result of that paper; [M] has utilised them in connection with rings of continuous functions; [S] has implicitly used them to answer a question of Semadeni.

With this background, we are ready for the main results. All spaces considered, are assumed to be Hausdorff.

§ 1

Theorem 1.1:

A) The following are equivalent for a separable space X:-

1) X is hereditarily locally compact.

2) X is locally compact and the set of accumulation points of X is discrete.

3) X is homeomorphic to $\psi(\underline{F})$ for some $\underline{F}$.

B) The following (4 through 7) are equivalent for a separable space X:-

4) X is pseudocompact and hereditarily locally compact.

5) X is a minimal h.l.c.s. space; that is, X is hereditarily locally compact and no strictly coarser Hausdorff topology is so.

6) X is homeomorphic to $\psi(\underline{F})$ for some maximal $\underline{F}$.

7) X is regular; the set of accumulation points of X is discrete; and every clopen discrete subset of X is finite.

Proof:

1) implies 2): Since X is separable, it has a countable dense subset Y; since X is h.l.c.s., this Y is locally compact. It follows by an easy application of Baire category theorem (or from known characterizations of countable locally compact spaces) that Y contains a discrete dense subset, say N. It is easily seen that every point of N has to be isolated in X and that there can be no other isolated points. Now suppose its complement $X \setminus N$ has an accumulation point x. Then the subspace $N \cup \{x\}$ is not open in its closure, namely X. Hence it cannot be locally compact. (For, in a locally compact space, a subspace is locally compact if and only if it is open in its closure). This contradicts 1).

2) implies 3): Let N be the set of all isolated points

of X. Then N must be contained in every dense subset of X. Since X is separable, it follows that N is countable. This N is dense in X, because the set $X \setminus N$ of accumulation points of X is assumed to be discrete. Now for each x in $X \setminus N$, let V_x be a compact neighbourhood of x such that $V_x \subset N \cup \{x\}$. We now claim that $\{V_x : x \in X \setminus N\}$ is a family of p.a.d. infinite subsets of N. For this, we first observe that each V_x is the set of a sequence in N converging to x. This is because every compact countable space with a unique accumulation point must be homeomorphic to $\omega+1$. If $V_x \cap V_y$ is infinite, then we get a sequence converging to both x and y; this is impossible in a Hausdorff space unless $x = y$. It is now clear that the topology of X is same as that of $\psi(\{V_x : x \in X \setminus N\})$.

3) implies 1): Clearly $(\underline{F})$ is locally compact. The pairwise almost disjointness of members of $\underline{F}$ assures the Hausdorffness of this space. If A is any subset of this space, the set $\bar{A} \setminus A$ is closed. (In fact, any subset of $X \setminus N$ is closed, since any subset containing N is open). Thus A is open in its closure and therefore locally compact.

4) implies 6): Since we have already proved that 1) implies 3), we may assume that X is $\psi(\underline{F})$ for some family $\underline{F}$ of p.a.d. infinite subsets of N. We have to deduce the maximality of $\underline{F}$ from the pseudocompactness of X. If $\underline{F}$ is not maximal, there is an infinite subset A of N. almost disjoint with every member of $\underline{F}$. Then every F^* (where $F \in \underline{F}$) has a neighbourhood disjoint with A. Thus A is infinite and clopen. Clearly then there is a continuous real function on X which maps A onto any countable set we please.

6) implies 7): X is locally compact and Hausdorff and therefore regular. The set of accumulation points of X is discrete, because every such point has a neighbourhood containing no other such point. If A is a clopen discrete subset of $\psi(\underline{F})$, then consider $A \cap N$; the maximality of $\underline{F}$ implies that either

$A\cap N$ is finite or $A\cap N\cap F$ is infinite for some F in $\underline{F}$; in the latter case F^* is a limit point of A, contrary to the assumption that F is closed and discrete; consequently, $A\cap N$ is finite; but since N is dense and A is open, we hawe $A\subset \overline{A\cap N}$, thereby proving that A its elf is finite.

7) implies 4): First, we shall prove that X is locally compact. At isolated points, the singletons form compact neighbourhoods. Let X be an accumulation on point. Then since the set of all accumulation points of X is discrete, there is a neighbourhood V_x of x such that x is the only accumulation point of V_x, inside V_x. By regularity, we may assume that this V_x is closed. If A is any infinite subset of $V_x\setminus\{x\}$, then A is open (since every point of A is isolated in X) and $A\cup\{x\}$ is closed (since $A\subset V_x = \overline{V}_x$ and since x is the only accumulation point of V_x); our assumption that everu clopen discrete subset of X is finite, therefore implies that x is a limit point of A. Thus x is a limit point of every infinite subset of V_x. Clearly then, V_x is compact.

Now the discreteness of the set of all accumulation points implies (as we have already seen) that every subset is locally closed (that is, closed in a bi-ger open set, or equivalenty open in its closure) and therefore locally compact.

In the space X, there is a dense subset (namely, the set of all isolated points) every infinite subset of which has a limit point in X (since every clopen discrete subset is finite). Hence if f is any real valued continuous function, the range $f(X)$ must contain a dense subset, every infinite subset of which has a limit point. In other words, some bounded subset is dense in $f(X)$. Clearly then $f(X)$ itself is bounded. Hence X is pseudo compact.

5) implies 6): By what we have proved already, 5) implies that X is $\psi(\underline{F})$ for some $\underline{F}$. If this $\underline{F}$ is not maximal, there is an infinite subset A of N which meats every member of $\underline{F}$

in a finite set. Now weaken the topology at the point F^* (for one fixed F in $\underline{F}$) by de claring that every neighbourhood by F^* must almost contain $F \cup A$. [That is, we consider $\psi(\underline{G})$, where $\underline{G} = (\underline{F}\setminus\{F\}) \cup \{F \cup A\}$] this is a coarser T_2 topology that is h.l.c.s.

6) implies 5): We claim that any coarser h.l.c.s. topology gives rise to a family $\underline{G}$ of p.a.d. infinite subsets of N and an one-to-one map θ from $\underline{G}$ to $\underline{F}$ such that G almost contains $\theta(G)$ for every G in $\underline{G}$. For each F in $\underline{F}$, take a compact neighbourhood G of F^* in the coarser topology, containing no other accumulation point in that topology; this is possible because that topology is also h.l.c.s.; $\underline{G}$ is the collection of all such G; the map θ is clear. If G and $\theta(G)$ are not almost equal for some G in $\underline{G}$, then $G \setminus \theta(G)$ is infinite, therefore meets some member F of $\underline{F}$ infinitely, therefore has F^* in its closure in both the topologies, therefore has both F^* and $(\theta(G))$ in its closure in the coarser topology, a contradiction to the choice of G.

Now $\underline{F}$ and $\underline{G}$ are two families of p.a.d. infinite subsets of N and θ is a bijection. (Reason: There cannot be non accumulation points in the coarser topology. For, in any h.l.c.s. space, the set of accumulation points is discrete; hence, if X is a new accumulation point and V_x is a compact neighbourhood of x containing no other accumulation point, the some infinite subset of V_x has some F^* in its closure, by maximality of $\underline{F}$; hence a contradiction) from $\underline{G}$ to $\underline{F}$ such that G and (G) are almost equal for each G in $\underline{G}$. It follows that the two topologies that we are considering, are identical.

<u>Corollaries 1.2</u>:

a) The following are equivalent for a separable space X:-
   1) X is hereditarily locally compact, pseudo compact, but not compact.
   2) X is minimal h.l.c.s. and uncountable.
   3) X is homeomorphic to $\psi(\underline{F})$ for some maximal and

infinite $\underline{F}$.

b) A space X is a subspace of an h.l.c.s. space if and only if either X itself is h.l.c.s. or X is the sum of an h.l.c.s. space with an uncountable discrete space of cardinality $\leq c$.

c) Every hereditarily separable hereditarily locally compact space is countable.

d) The class of pseudo-compact h.l.c.s. spaces is stable under the formation of finite sums, clopen subspaces and quotients that are one-to-one except on a finite subset of the domain.

e) Every h.l.c.s. space has an h.l.c.s. pseudo compact extension.

Remarks 1.3:

a) The assumption of separability in Theorem 1.1. cannot be deleted. The one-point-compactification of an uncountable discrete space satisfies 1), 2), 4), 5) and 7), but not 3) and 6) of the Theorem. There may even exist (we do not know) spaces satisfying 1) but not 2).

b) The following can now be easily proved: A noncompact T_3 space is $\psi(\underline{F})$ for some maximal $\underline{F}$ if and only if there is a subset D such that i) D is countable, ii) D is open, iii) both D and its complement are discrete and iv) every sequence in D has a subsequence converging in X.

c) Proposition: A space X is a continuous image of a pseudo-compact h.l.c.s. space if and only if X contains a countable dense set D such that every sequence in D has a subsequence convergent in X.

To prove the 'if' part, take a maximal family $\underline{F}$ of p.a.d. infinite subsets of D that are sets of convergent sequences in X. Our asumption implies that if A is any infinite subset of D, then there is an infinite subset B of A that is the set of a convergent sequence; $B \cap F$ is infinite for some F in $\underline{F}$; hence $A \cap F$ is infinite. Thus $\underline{F}$ is a maximal family of p.a.d. infinite subsets of X (without any further condition on its members). Clearly X is the image of $\psi(\underline{F})$ under the abvious continuous map.

d) It can be proved that the following are equivalent for an h.l.c.s. space: i) -compactness, ii) Lindelofness, iii) hereditarily separability iv) metrizability v) the first countability of its one-point- compactification, vi) normality, vii) second countability and viii) countability.

§ 2

We say that a space is of type ψ is it is homeomorphic to $\psi(\underline{F})$ for some maximal $\underline{F}$; we say that it is of type ψ^* if it is homeomorphic to the one-point-compactification of some space of type ψ.

Theorem 2.1.: A topological space X is of type ψ^* if and only if it satisfies the following four conditions:-

i) X is compact Hausdorff.

ii) X is separable.

iii) The set of accumulation points of X has a unique accumulation point x_o.

and iv) No sequence of isolated points of X converges to x_o.

Proof: If X is a space of type ψ^*, let x_o be its point such that $X \setminus \{x_o\}$ is of type ψ. Then $X \setminus \{x_o\}$ satisfies the conditions of Theorem 1.1 and is in particular separable; hence X is separable. Clearly, X is compact and Hausdorff. To prove iii) we observe that the set of accumulation points of $X \setminus \{x_o\}$ is infinite and discrete and hence cannot be closed in X, whereas it is closed in $X \setminus \{x_o\}$. To prove iv) let B be the set of a sequence of distinct isolated points. Then by the maximality of

$\underline{F}$, (where $\underline{F}$ is the family such taht $X \setminus \{x_o\}$ is homeomorphic to $\psi(\underline{F})$, there is F in $\underline{F}$ such that $B \cap F$ is infinite. (A set is not distinguishad by us from its image under the above homeomorphism and hence this is meaningful.) Then F^* is in the closure of B. Hence the sequence that we started with, cannot converge to any point other then F^*. In particular, it can not converge to x_o.

Conversely, let X be a space satisfying these four conditions. Let Y be the subspace $X \setminus \{x_o\}$. Then clearly Y is locally compact, Hausdorff, non-compact and separable. Further, the set of all accumulation points of Y is discrete. If W is a clopen discrete infinite subset of Y, them $W \cup \{x_o\}$ will be closed in X and hence compact; since every point of W has to be isolated, this means that W is the set of a sequence of isolated points of X converging to x_o. This contradicts iv), Thereby proving that every clopen discrete subset of Y is finite. Thus Y satisfies the condition 7) of Theorem 1.1 and therefore is of type ψ. Thus X is of type ψ^*.

<u>Theorem 2.2</u>: Let X be any space such that

1) It is locally compact and Hausdorff.

2) It is scattered, with derived langth 2.

and 3) It is sequential, with sequential order 2.

Then X contains a subspace of type ψ^*. (Conversely, it is easy to prove that any space of type ψ^* satisfies these three conditions). (See [K] for the definitions of some new terms here).

<u>Proof</u>: Let x_o be a point in X with sequential order 2. Let W be a compact open neighbourhood of x_o containing no other point of derived length ≥ 2, Since the sequential order at x_o is 2, there is a subset A of W such that x_o is in $\bar{A}$, but no sequence from A converges to x_o. Let B the set of those points of A that are accumulation points of X. We claim that x_o is not in $\bar{B}$. We observe that $\bar{B} \cup \{x_o\}$ has at

most one accumulation point and hence Frechet. (Since it is already sequential). Therefore, if x_o were in $\overline{B}$, there would be a sequence from B (and hence from A) converging to x_o, contradicting our choice of A. Thus $x_o \notin \overline{B}$ and hence we may as well assume that every point of A is isolated in X, Since every sequential space is countably generated, we can choose a countable subset $\overline{C}$ of A such that $x_o \in \overline{C}$. Now let $Y = \overline{C}$. Then the space Y satisfies the four conditions of Theorem 2.1.; therefore it is of type ψ^*.

Remarks:

a) The spaces of type ψ^* are thus test spaces to verify Frechet property, among a fairly good class of spaces. A (locally) compact space is said to be of type (2,1) if there is a unique point in its second derived set. (This is a standard terminology, introduced by sierpinski). A restatement of our theorem reads like this: A compact Hausdorff sequential space of type (2,1) fails to be Frechet if and only if it contains an uncountable minimal h.l.c.s. subspace. In this connection, the equivalence of the following three assertions, is also an easy consequence of our results, for a separable compact Hausdorff space X of type (2,1).

i) X is not first-countable.
ii) X contains an uncountable discrete subspace.
iii) X contains an uncountable h.l.c.s. subspace.

b(It will be interesting to know whether 2) can be deleted in Theorem 2.1.

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1) Madurai University,
Madurai 625021,
INDIA.

2) Universidad de Los Andes
VENEZUELA.

VK/MR/codea