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# "USES OF $\beta S$ IN INVARIANT MEANS AND EXTREMELY LEFT AMENABLE SEMIGROUPS"

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# "USES OF βS IN INVARIANT MEANS AND EXTREMELY LEFT AMENABLE SEMIGROUPS"

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Abstract: A semigroup S is called extremely left amenable if there is a multiplicative left invariant mean on the space m(S) of all bounded real valued functions on S. A subset $A \subset S$ is called left thick if given a finite subset $B \subset S$ there is an element 'a' in A so that $Ba \subset A$. We prove the following in this paper:

The following are equivalent for a semigroup S:

(a) S is extremely left amenable.

(b) There is an ultrafilter on S so that all members of this ultrafilter are left thick.

(c) The collection of all left thick subsets of S can be expressed as a union of ultrafilters.

We further show that the set of all multiplicative left invariant means on S is either finite or has cardinality greater than or equal to $2^c$. If a semigroup has a unique multiplicative left invariant mean then it is also extremely right amenable. We investigate the relationship between continuous extensions of the semigroup operation in S to βS and extreme amenability and when will the collection of left thick subsets of S form a filter.

**INTRODUCTION:** One of the important problems in mathematics is to find when will a given collection of self maps of a set $X$ have a common fixed point. This led to the study of existence of invariant means on certain function spaces associated with a semigroup $A$. We mention in this connection, the works of Day [5]; Rickert [20]; Furstenberg [7]; A. T. Lau [22], among many others. In 1966; Mitchell [17,18], introduced the class of semigroups which have the fixed point property on compacta and showed that they coincide with the class of semigroups $S$ which admit a multiplicative left invariant mean on the space $m(S)$ of bounded real valued functions on $S$. Granirer [9] gave algebraic characterization of extremely left amenable semigroups and proved that it is a large class of semigroups. So far no topological characterization of extreme amenability of a semigroup in known. In this paper we study the extreme left (or right) amenability of a semigroups $S$ from its Stone-Čech compactification $\beta S$ or equivalently in terms of ultrafilters on $S$. This point of view gives us two dividends. First of all we are able to study the cardinality of the set of all multiplicative left invariant means on $S$. We remark that the parallel study of the dimension of the set of all left invariant means on a semigroup $S$ was a difficult problem and formed the subject matter of investigation of the doctoral dissertations of I. S. Luthar [14] and E. Granirer [8]. In another direction we get the interesting result that if $S$ is an Arens regular semigroup or a weakly almost periodic semigroup then the uniqueness of multiplicative left invariant means on $S$ implies that $S$ is uniquely extremely right amenable as well and that $S$ has a unique multiplicative two sided invariant mean.

Notations: For purely topological notions and Stone-Čech compactifications of completely regular spaces we follow [21]. The notions on Banach spaces are as in [13]. $S$ denotes a semigroup with discrete topology. $m(S)$ denotes the Banach space of all bounded real valued functions on $S$ with usual vector space operations and supremum norm. If $X$ is a Hausdorff, completely regular space then $\beta X$ denotes its Stone-Čech compactification. If $X$ is a $T_2$, completely regular space, $\beta X$ its Stone-Čech compactification, $Y$ a compact $T_2$ space $f : X \to Y$ a continuous function then $\tilde{f} : \beta X \to Y$ denotes the unique continuous - extension of $f$ to $\beta X$. If $a \in S$ then $\ell_a : S \to S$ denotes the map $x \to ax$ on $S$ and $r_a$ denotes the map $x \to xa$ on $S$. $\tilde{\ell}_a$, $\tilde{r}_a$ are the unique continuous extensions of $\ell_a$ and $r_a$ respectively to maps from $\beta S$ into itself. We put $\tilde{\ell}_a(x) = ax$ for all $x \in \beta S$ and $a \in S$. If $E$ is a Banach space $E^*$ denotes its dual. Given $S$ and a subset $A \subset S$ we denote by $\phi_A$ or $\chi_A$ the characteristic function of $A$. Thus $\phi_A(x) = 1$ if $x \in A$ and $\phi_A(x) = 0$ if $x \in S - A$. If $\mu \in (m(S))^*$ then $\mu(A)$ denotes $\mu(\phi_A)$. The constant function $\phi_S$ is denoted by $1$.

Definition 1.1: Let $a \in S$ and $f$ a complex valued function on $S$. We denote by $\ell_a f$ (similarly $r_a f$) the function $f \circ \ell_a (f \circ r_a)$ on $S$. The map $f \to \ell_a f$ $(f \to r_a f)$ on $m(S)$ defined by an element $a \in S$ is also denoted by $\ell_a(r_a)$. This will not cause confusion with the earlier meaning of $\ell_a(r_a)$.

$\ell_a^*(r_a^*)$ denotes the adjoint operator of $\ell_a(r_a)$ on $(m(S))^*$ for all $a \in S$.

A mean on $S$ is a positive linear functional $\mu$ on $m(S)$ so that $\mu(\phi_S) = 1$. A mean $\mu$ on $S$ is called left invariant (right invariant; two sided invariant) if $\ell_a^*\mu = \mu (r_a^*\mu = \mu$, $\ell_a^*\mu = \mu = r_a^*\mu)$ for all $a \in S$. A semigroup $S$ is called left (right, two-sided) amenable if there exists a left (right, two-sided) invariant mean on $S$.

<u>Definition 1.2</u>: A multiplicative mean on $S$ is a mean $\mu$ on $S$ so that $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in m(S)$. A semigroup $S$ is called extremely left (right, two-sided) amenable if there is a multiplicative left ( right, two-sided) invariant mean on $S$.

<u>Definition 1.3</u>: A subset $A \subset S$ of a semigroup $S$ is called left thick (right thick) if given a finite set $a_1, a_2, \ldots, a_n$ of elements of $S$ there is an element $g \in A$ so that $a_i g \in A (g a_i \in A)$ for all $i = 1,2,3,\ldots,n$. The subset $A$ is called a left ideal (right ideal, two-sided ideal) if $xa \in A (ax \in A$, both $xa$ and $ax \in A)$ for all $x \in S$ and $a \in A$.

The idea for left thick sets and extremely left amenable semigroups were introduced by T. Mitchell [17]. Algebraic characterizations of left extreme amenability are given in a nice paper of E. Granirer [9]. We approach the problem of characterizing extreme amenability from topological point of view. For this we investigate how to find means on $S$. The easiest means on $S$ are evaluations at points of $\beta S$. Thus it is important to know when will a mean defined by evaluation at an element of $\beta S$ is left or right invariant. This leads us to consider the extended multiplication $\tilde{\ell}_a$ on $\beta S$ for all $a \in S$ and finding out when will $\beta S$ have a right zero. This brings us back to viewing the non isolated elements of $\beta S$ as free ultrafilters and studing the existence of special ultrafilters on $S$. So we give below some properties of $\beta S$ which we need in the future.

Definition 1.4: Let $S$ be a semigroup with discrete topology. Let $a \in \beta S$. Then $\varepsilon_a$ is the mean defined on $S$ by the formula $\varepsilon_a(f) = \tilde{f}(a)$ for all $f \in m(S)$.

The following is an easy consequence of the fact that every bounded complex valued function $f$ on $S$ extends continuously to $\beta S$ and a theorem on characterization of multiplicative - linear functionals on the space of all continuous complex-valued functions on a compact Hausdorff space proved in [13].

Lemma 1.5: Every multiplicative mean on $S$ is of the form $\varepsilon_a$ for some $a \in \beta S$.

Lemma 1.6: Let $a \in \beta S$ and $V_a = \{V \cap S \mid V$ is a neighborhood of a in $\beta S.\}$. Then $V_a$ is an ultrafilter in $S$ and $\{a\} = \bigcap_{W \in V_a} \overline{W}$. Every ultrafilter $\mathcal{U}$ on $S$ is of the form $V_a$ for some $a \in \beta S$ and $\mathcal{U}$ is free if and only if $\mathcal{U} = V_a$ for some $a \in \beta S/S$. Further, let $a \in \beta S$ and $E \subset S$. Then $E \in V_a$ if only if $a \in \overline{E}$.

Proof: Lemma 1.6 is well-known and can be seen for example in [4].

Now we note that if $a \in \beta S$ then $\varepsilon_a(f) = \tilde{f}(a)$ for all $f \in m(S)$. Thus $\varepsilon_a$ is left invariant if and only if $\varepsilon_a(f) = \varepsilon_a(\ell_b(f))$ for all $f \in m(S)$ and $b \in S$. Now $\varepsilon_a(f) = (\tilde{f})(a)$ and $\varepsilon_a(\ell_b f) = (\widetilde{\ell_b f})(a) = \tilde{f}(ba)$ for all $b \in S$ and $f \in m(S)$. So $\varepsilon_a(f) = \varepsilon_a(\ell_b f) \Leftrightarrow \tilde{f}(a) = f(ba)$. Combining this with lemma 1.5 we get:

Lemma 1.7: A semigroup $S$ has a multiplicative left invariant mean, or in other words $S$ is extremely left amenable if and only if there is an element $a \in \beta S$ so that $ba = a$ for all $b \in S$. Likewise $S$ is extremely right amenable if and only if there is an element $a \in \beta S$ so that $ab = a$ for all $b \in S$.

Definition 1.8: Let $S$ be a semigroup. An element $a \in \beta S$ is called a left zero (right zero) of $\beta S$ if and only if $ab = a$ ($ba = a$) for all $b \in S$. An element of $\beta S$ is said to be a zero of $\beta S$ if and only if it is both a right zero and left zero of $\beta S$.

The lemma 1.7 can be restated as:

Corollary 1.9: A semigroup $S$ is extremely left (right) amenable if and only if $\beta S$ has a right (left) zero. $\beta S$ has a two sided zero if and only if $S$ is two-sided extremely amenable.

Now if $b \in S$ then $\tilde{\ell}_b : \beta S \to \beta S$ is a continuous function from $\beta S$ into $\beta S$. So if $a \in \beta S$ is a right zero of $\beta S$ and $b_1, b_2, \ldots, b_n \in S$ then we have that given a neighborhood $W$ of a in $\beta S$ there are neighborhoods $V_1, V_2, \ldots, V_n$ of a in $\beta S$ so that $\tilde{\ell}_{b_i}(V_i) \subset W$ for $i = 1, 2, \ldots, n$. Then choosing an element $x \in W \cap V_1 \cap \ldots \cap V_n$ we get that $b_i x \in W \cap S$ for all $i = 1, 2, \ldots, n$. So $W \cap S$ is left thick in $S$ and also is a member of the ultrafilter $V_a$. Thus an easy application of lemma 1.6 gives us:

<u>Lemma 1.10</u>: Let $S$ be extremely left amenable. Then there is an ultrafilter Ⓤ in $S$ so that all members of Ⓤ are left thick in $S$. A similar result holds for extremely right or two-sided amenable instead of extremely left amenable.

Note that lemma 1.10 and lemma 1.6 give that if $S$ is extremely left amenable then there is an element $a \in \beta S$ and a multiplicative left invariant mean $\mu$ on $S$ so that $E$ is left thick and $\mu(E) = 1$ for all $E \in V_a$. (Take $\mu = \varepsilon_a$).

The converse also holds as is shown in the following lemma.

<u>Lemma 1.11</u>: Let $S$ be an extremely left amenable semigroup. Let $E \subset S$ be left thick. Then there exists a multiplicative left invariant mean $\theta$ on $S$ so that $\theta(\phi_E) = 1$. Moreover there is an ultrafilter Ⓤ on $S$ so that $E \in$ Ⓤ and all members of Ⓤ are left thick subsets of $S$. A similar theorem holds if left is replaced by right troughout in the lemma.

<u>Proof</u>: Let $a \in \beta S$ be a right zero of $\beta S$. For each finite non

empty set $F \subset S$ let $x_F$ be an element of $E$ so that $Yx_F \in E$ for all $y \in F$. Let $b$ be a cluster point of the net $x_F$ when $F$ varies over the filter of all finite subsets of $S$ under containment. Let $r_b : S \to \beta S$ be the map $r_b(x) = xb$ for all $x \in S$ and let $\tilde{r}_b$ be its unique continuous extension to $\beta S$. Let $z = \tilde{r}_b(a)$. Then $z \in \bar{E}$. To see this let $Y_\alpha$ be a net in $S$ converging to 'a'. For a fixed $\alpha$, $Y_\alpha x_F \in E$ eventually when $F$ varies over the filter of all finite subsets of $S$. So $\tilde{r}_b(Y_\alpha) = Y_\alpha b = \text{lt}\ Y_\alpha x_F$ belongs to $\bar{E}$. Now $Y_\alpha \to a$. So $z = \tilde{r}_b(a) = \text{lt}\ \tilde{r}_b(Y_\alpha) \in \bar{E}$. Now $\varepsilon_z$ is left invariant. To see this let $f \in m(S)$ and $t \in S$. Now $\varepsilon_z(f) = \tilde{f}(z)$ and $\varepsilon_z(\ell_t f) = \tilde{f}(tz)$. So it is enough to show that $tz = z$. Now $z = \tilde{r}_b(a) = \text{lt}\ \tilde{r}_b(Y_\alpha) = \text{lt}\ Y_\alpha b$ for some net $Y_\alpha \to a$. So

11 $tz = \text{lt}\, tY_\alpha b = \tilde{r}_b(\text{lt}(tY_\alpha)) = \tilde{r}_b(ta) = \tilde{r}_b(a) = z$ since $ta = a$ for all $t \in S$. Put $\theta = \varepsilon_z$. Since $z \in \bar{E}$ we get that $E \in V_z$ and $\theta(\phi_E) = 1$ by lemma 1.6. Moreover $V_z$ is an ultrafilter all of whose members are left thick. Thus we get the lemma in the left invariant case. The right invariant case is proved - similarly.

T. Mitchell [18] proved that a subset $E \subset S$ of a left amenable semigroup $S$ is left thick if and only if there is a left invariant mean $\mu$ on $S$ so that $\mu(\phi_E) = 1$. We note that lemma 1.11 is an analogue of this theorem of Mitchell in the multiplicative case.

We also get most part of the following characterization of extreme left amenability by an easy application of lemma 1.11, - 1.10 and 1.6.

<u>Theorem 1.12</u>: Let S be a semigroup. Then the following are equivalent:

(a) There is at least one ultrafilter Ⓤ on S all of whose members are left thick subsets of S.

(b) The collection Ⓦ of all left thick subsets of S can be expressed as a union of ultrafilters of S.

(c) If A, B are subsets of S then $A \cup B$ is left thick if and only if at least one of the sets A or B is left thick.

(d) $\beta S$ has a right zero.

(e) S is extremely left amenable.

<u>Proof</u>: The equivalence of (d) and (e) has been proved in - corollary 1.9. The implication (e) $\Rightarrow$ (a) is lemma 1.10. To see that (a) $\Rightarrow$ (e) let Ⓤ be an ultrafilter in S all of whose members are left thick subsets of S. Let $a \in \beta S$ be such that Ⓤ $= V_a$ as in lemma 1.6. Let $b \in S$. It is enough to show that $ba = a$. For that it is enough to show that $ba \in \overline{E}$ where $E \in$ Ⓤ. Now given $F \in$ Ⓤ, choose $x_F \in E \cap F$ so that $bx_F \in E \cap F$. (This can be done because $E \cap F \in$ Ⓤ and each member of Ⓤ is left thick.) If t is a cluster point of the net $(x_F)$ then $t \in \overline{F}$ for all $F \in$ Ⓤ. So

$t = a$ by lemma 1.6. So $x_F \to a$. Likewise $bx_F \to a$. However, the continuity of $\tilde{\ell}_D$ gives that $bx_F \to ba$. So $ba = a$. So we get (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). It is easy to see that (b) $\Rightarrow$ (a). To see that (a) $\Rightarrow$ (b), apply the equivalence (a) $\Leftrightarrow$ (e) and lemma 1.11. So, we get (e) $\Leftrightarrow$ (d) $\Leftrightarrow$ (a) $\Leftrightarrow$ (b). We now prove (b) $\Rightarrow$ (c). Assume (b) and that $A$, $B$ are subsets of $S$ and $A \cup B$ is left thick. Now $A \cup B$ belongs to an ultrafilter Ⓥ all of whose members are left thick from (b). Since Ⓥ is ultrafilter, it follows that either $A$ or $B$ belongs to Ⓥ. So either $A$ or $B$ is left thick. So (b) $\Rightarrow$ (c). To prove (c) $\Rightarrow$ (b) assume (c). Let Ⓛ be the collection of all left thick subsets of $S$. Let $E \in$ Ⓛ. Apply Zorn's lemma to the collection of all filters Ⓕ so that $E \in$ Ⓕ and Ⓕ $\subset$ Ⓛ. Then we get a filter Ⓗ so that the following hold:

(i) Ⓗ $\subset$ Ⓛ

(ii) $E \in$ Ⓗ

(iii) Ⓗ is a filter

(iv) Ⓗ is maximal with respect to the properties (i), (ii) and (iii) above.

We claim that Ⓗ is an ultrafilter of subsets of $S$. For this it is enough to show that if $A \subset S$ then either $A$ or $S/A$ belongs to Ⓗ. If $A \cap M$ is left thick for all $M \in$ Ⓗ then it follows that $A \in$ Ⓛ and $A \cap M \neq \phi$ and left thick and belongs to Ⓗ for all $M \in$ Ⓗ. So the maximality of Ⓗ implies that $A \in$ Ⓗ. Suppose that there is a set $K \in$ Ⓗ so that $A \cap K$ is not left thick. Let $M \in$ Ⓗ. Then $A \cap K \cap M$ is not left thick. However, we have $K \cap M$ is left thick. So an application of (c) gives us that $(S/A) \cap K \cap M$ is left thick for all $M \in$ Ⓗ. This in turn implies that $(S/A) \cap M$ is left thick for all $M \in$ Ⓗ and hence $S/A \in$ Ⓗ. So we get that (c) $\Rightarrow$ (b). Thus we have the theorem.

Finally we remark that the statements in theorem 1.12 hold when left is replaced by right in that theorem 1.12 and can be probed in a similar fashion.

Now we come to the cardinality problems related to multiplicative invariant means.

## SECTION 2. CARDINALITY OF MULTIPLICATIVE INVARIANT MEANS.

In this section we study the cardinality of multiplicative left invariant means. An analoguous problem of uniqueness and dimensionality of left invariant means on a left amenable - semigroup was considered to be an important problem and formed the subject matter of investigations by I. S. Luthar [13] and E. Granirer [8]. Our results on the multiplicative invariant means are sharp. We get the surprising result that if a semigroup has a unique multiplicative left invariant mean then it has also got a multiplicative right invariant mean.

We need the following results for future discussion. If $A$ is a set let $|A|$ denote its cardinality.

<u>Lemma. 2.1</u>: Let $X$ be a discrete space and $\beta X$ its Stone-Čech compactification. Then every closed subset $F \subset \beta X$ is either finite or $|F| \geq 2^c$ where $c$ is the cardinality of continuum. For a proof see [19], [6] or [4].

<u>Lemma 2.2</u>: Let $S$ be a semigroup. Let $a,b,c \in \beta S$ and at least two among $a,b,c$ be in $S$. Then all the products $(ab)$, $(bc)$, $a(bc)$, $(ab)c$ are defined and $(ab)c = a(bc)$.

<u>Proof</u>: We prove it only in the case when $a,c \in S$. The other cases are proved similarly. Let $b_\alpha$ be a net in $S$ so that $b_\alpha \to b$. Then $(ab) = \tilde{\ell}_a b = \text{lt}\ ab_\alpha$. Moreover

$$(ab)c = \tilde{r}_c(ab) = \text{lt}_{\alpha}(ab_\alpha)c = \text{lt}_{\alpha}\ a(b_\alpha c) = \tilde{\ell}_a(\text{lt}(b_\alpha c)) = \tilde{\ell}_a(\tilde{r}_c(b))$$

$$= \tilde{\ell}_a(bc) = a(bc).$$

The corollary 1.9 gave the relationship between a left zero of $\beta S$ and fixed points of the maps $\tilde{r}_b$ where $b \in S$. Motivated by this we put:

Definition 2.3: Let $S$ be a semigroup and $b \in S$. Define the left fixed point set $L_b$ of $b$ to be the set $\{x \mid x \in \beta S$ and $bx = x\}$. Define $R_b$ the right fixed point set of $b$, to be the set $\{x \mid x \in \beta S$ and $xb = x\}$.

The lemmas 1.5 and 1.7 give us that the set of all multiplicative right invariant means on $S$ is exactly $\bigcap_{b \in S} L_b$ and the set of multiplicative left invariant means on $S$ is $\bigcap_{b \in S} R_b$. Since each $L_b$ and $R_b$ is closed in $S$ we get using, lemma 2.1, the following.

Theorem 2.4: The set of all multiplicative left invariant means as well as the set of all multiplicative right invariant means are closed subsets of $\beta S$. Hence a semigroup $S$ can have either only finitely many multiplicative left invariant means or has at least $2^c$ multiplicative left invariant means. A similar result holds for the set of multiplicative right invariant means.

Note that the simplest of all infinite cardinals namely $\aleph_0$ does not appear as a value in many natural cardinality problems in functional analysis. The theorem 2.4 is one instance where we see that $\aleph_0$ is not the cardinality of the set of multiplicative invariant means on any semigroup $S$. Analoguously Bhaskara Rao [3] proved that the cardinality $m$ of a Banach spaces satisfies the equation $m^{\aleph_0} = m$ and S. Janakiraman and M. Rajagopalan [12] proved that there is no interval of cardinality $\aleph_0$ of locally compact group topologies on any Abelian group.

It would be interesting to know what cardinals can appear as the cardinals of the set of all multiplicative left invariant means on a semigroup S.

It is easy to see that every integer $n \geq o$ as well as $2^c$ can be such cardinalities due to the following examples:

Example 2.5: Let Z be the set of all integers under usual addition. Then Z has no multiplicative left invariant mean.

Proof: This follows from the algebraic characterization of extreme left amenability due to Granirer [9]. Granirer showed that a semigroup S is extremely left amenable if and only if given $x, y \in S$ there is an $a \in S$ so that $xa = ya = a$. This obviously is not satisfied by Z.

Example 2.6: Let $n > o$ be a given integer. Let X be a set with n elements with right zero multiplication. That is $ab = b$ for all $a,b \in X$. Then clearly $\beta X = X$ and each element of X is a right zero of X. So X has exactly n multiplicative left invariant means.

Example 2.7: Let N be the set of integers $> o$ with right zero multiplication. Thus $ab = b$ for all $a,b \in N$. Then it is easily checked that $ax = x$ for all $a \in N$ and $x \in \beta N$. Since $|\beta N| = 2^c$ (see [18]) we have that $2^c$ is a possible value of the cardinality set of all multiplicative left invariant means on a semigroup.

We remark that if $n > 1$ in example 2.6 then that semigroup $X$ is not even right amenable though $X$ is extremely left - amenable. So it is interesting that this cannot happen if the multiplicative left invariant mean is unique as the following theorem shows:

Theorem 2.8: Let $S$ be a semigroup. Let $S$ have a unique multiplicative left invariant mean. Then $S$ is also extremely right amenable.

Proof: Let $a \in \beta S$ be a right zero of $\beta S$. Let $b, d \in S$. - Then $b(ad) = (ba)d$ by lemma 2.2. so $b(ad) = (ba)d = (ad)$. So $ad$ is also a right zero of $\beta S$. The uniqueness of multiplicative left invariant mean gives that $a = ad$. Since $d \in S$ is arbitrary it follows that 'a' is also a left zero of $\beta S$. So $\varepsilon_a$ is also a multiplicative right invariant mean on $S$. Thus we get the theorem.

The above theorem does not give that if $S$ has a unique multiplicative left invariant mean then it also has a unique multiplicative right invariant mean. It would be interesting to know whether a semigroup is uniquely extremely left amenable if and only if it is uniquely extremely right amenable. We can settle this problem in one particular case in the affirmative. But then we have to consider continuous extensions of the - semigroup operation of $S$ to $\beta S$ and their relation to extreme amenability.

Theorem 2.9: Let $S$ be a semigroup. Then there is a binary operation $\odot_\ell$ on $\beta S$ so that the following holds:

(a) $\beta S$ is a semigroup under $\odot_\ell$.

(b) $S$ is a subsemigroup of $\beta S$ under $\odot_\ell$ and $\odot_\ell$ agrees with the given operation on $S$.

(c) $x \odot_\ell Y$ is continuous in $Y$ on $S$ for any fixed $x$ of $\beta S$.

Similarly an extension $\odot_r$ of the semigroup operation on $S$ can be defined in $\beta S$ which is continuous in the left variable only.

Proof: For $x \in S$ let $r_x$ be the map $Y \to Yx$ on $S$ and let $\tilde{r}_x$ be the continuous extension of $r_x$ to $\beta S$. If $a \in \beta S$ let $\ell_a$ be the map $x \to \tilde{r}_x(a) = ax$ for all $x \in S$. Then $\ell_a$ is a continuous map from $S$ into $\beta S$ and hence has a unique - continuous extension $\tilde{\ell}_a$ to $\beta S$. Finally if $a, b \in \beta S$ put $a \odot_\ell b = \tilde{\ell}_a(b)$. Then it is clear from the definition of $\tilde{\ell}_a$ that $a \odot b$ is continuous in $b$ for all $a \in \beta S$. Let $a,b,c, \in \beta S$.

Let $b_\alpha$ be a net in $S$ converging to $b$ and $c_\beta$ a net in $S$ converging to $c$ in $\beta S$. Now $(b \odot_\ell c) = \tilde{\ell}_b(c) = \underset{\beta}{\text{lt}}\, bc_\beta$. So $a \odot_\ell (b \odot_\ell c) = \underset{\beta}{\text{lt}}\, a \odot_\ell (bc_\beta)$. Now $c_\beta \in S$, and $b_\alpha \to b$. So $bc = \underset{\alpha}{\text{lt}}(b_\alpha c_\beta)$. So $a \odot_\ell (b \odot_\ell c) = \underset{\beta}{\text{lt}}[a \odot_\ell \underset{\alpha}{\text{lt}}(b_\alpha c_\beta)] = \underset{\beta}{\text{lt}}[\underset{\alpha}{\text{lt}}(ab_\alpha c_\beta)]$ $= \underset{\beta}{\text{lt}}[(\underset{\alpha}{\text{lt}}(ab_\alpha))c_\beta]$ since $c_\beta \in S$ $= \underset{\beta}{\text{lt}}[(a \odot_\ell b)c_\beta]$ from definition of $a \odot_\ell b = (a \odot_\ell b) \odot_\ell c$. So $\beta S$ is a semigroup under $\odot_\ell$. It operation $\odot_\ell$ on $\beta S$ so that the following holds:

(a) $\beta S$ is a semigroup under $\odot_\ell$.

(b) $S$ is a subsemigroup of $\beta S$ under $\odot_\ell$ and $\odot_\ell$ agrees with the given operation on $S$.

(c) $x \odot_\ell Y$ is continuous in $Y$ on $S$ for any fixed $x$ of $\beta S$.

Similarly an extension $\odot_r$ of the semigroup operation on $S$ can be defined in $\beta S$ which is continuous in the left variable only.

<u>Proof</u>: For $x \in S$ let $r_x$ be the map $Y \to Yx$ on $S$ and let $\tilde{r}_x$ be the continuous extension of $r_x$ to $\beta S$. If $a \in \beta S$ let $\ell_a$ be the map $x \to \tilde{r}_x(a) = ax$ for all $x \in S$. Then $\ell_a$ is a continuous map from $S$ into $\beta S$ and hence has a unique continuous extension $\tilde{\ell}_a$ to $\beta S$. Finally if $a, b \in \beta S$ put $a \odot_\ell b = \tilde{\ell}_a(b)$. Then it is clear from the definition of $\tilde{\ell}_a$ that $a \odot b$ is continuous in $b$ for all $a \in \beta S$. Let $a,b,c, \in \beta S$.

Let $b_\alpha$ be a net in $S$ converging to $b$ and $c_\beta$ a net in $S$ converging to $c$ in $\beta S$. Now $(b \odot_\ell c) = \tilde{\ell}_b(c) = \lim_\beta bc_\beta$. So $a \odot_\ell (b \odot_\ell c) = \lim_\beta a \odot_\ell (bc_\beta)$. Now $c_\beta \in S$, and $b_\alpha \to b$. So $bc = \lim_\alpha (b_\alpha c_\beta)$. So $a \odot_\ell (b \odot_\ell c) = \lim_\beta [a \odot_\ell \lim_\alpha (b_\alpha c_\beta)] = \lim_\beta [\lim_\alpha (ab_\alpha c_\beta)]$ $= \lim_\beta [(\lim_\alpha (ab_\alpha))c_\beta]$ since $c_\beta \in S$ $= \lim_\beta [(a \odot_\ell b)c_\beta]$ from definition of $a \odot_\ell b = (a \odot_\ell b) \odot_\ell c$. So $\beta S$ is a semigroup under $\odot_\ell$. It is clear that $a \odot_{\ell} b = ab$ for all $a, b \in S$. So we get the theorem.

The use of $\odot_{\ell}$ in theory of numbers is given in [4]. It is not true in general that if S is a semigroup then the multiplication in S extends to a semigroup operation in $\beta S$ which is continuous in each variable separately. It is needless to say that the multiplication need not necessarily extend to a jointly continuous multiplication in $\beta S$. Continuous extensions of multiplications from S to $\beta S$ is a very fascinating and hard study and some partial results have been obtained by H. Mankowitz [15], T. Macri [16]; R. P. Hunter and L. W. Anderson [1], Aravamudan [2], and others. The following gives us an interesting class of semigroup:

Definition 2.10: A semigroup S is called an R-semigroup if the multiplication in S extends jointly continuously to a semigroup operation in $\beta S$. The semigroup S is called a V-semigroup if the multiplication in S extends to a semigroup operation in $\beta S$ which is continuous in each variable separately. Now we are ready to improve our theorem 2.9 for the class of V-semigroups.

Theorem 2.11: Let S be a semigroup. Then the following are equivalent:

(i) S has a unique multiplicative left invariant mean.

(ii) The collection of all left thick subsets of S is an ultrafilter in S.

(iii) $\beta S$ has a unique right zero under the operation $\odot_r$.

(iv) Given $f \in m(S)$ there exists a unique constant function in $k(f)$ where $k(f)$ is the weak$^*$ - closure of the set $\{r_a f \mid a \in S\}$.

A similar theorem holds if left is interchanged with right throughout.

Proof: The equivalence (i)$\Leftrightarrow$(ii) follows from theorem 1.12. The equivalence of (ii) and (iii) follows from theorem 1.12 and lemmas 1.7 and 1.10. We now show (iv) $\Rightarrow$(iii). To see that observe that if $a \in \beta S$ is a right zero of $\beta S$ in $\odot_r$ and $a_\alpha$ is a net in $S$ which converges to $a$ in $\beta S$ then $r_{a_\alpha} f$ converges in weak$^*$ topology to the constant function $\tilde{f}(a)$. Thus we have that given $f \in m(S)$ and a right zero 'a' of $\beta S$ the constant function $\tilde{f}(a)$ belongs to $k(f)$. So it is clear that iv) $\Rightarrow$(iii). The implication (iii) $\Rightarrow$ (iv) follows easily by using theorem 1.1 of [10]. Thus we have the theorem.

Theorem 2.12: Let $S$ be a semigroup. If $S$ is a V-semigroup then the operations $\odot_\ell$ and $\odot_r$ coincide on $\beta S$. Conversely if $\odot_\ell$ coincides with $\odot_r$ on $\beta S$ then $S$ is a V-semigroup.

The proof is straightforward and hence is omitted.

Theorem 2.13: Let $S$ be a V-semigroup. Then the following are equivalent:

(1) $S$ has a unique multiplicative left invariant mean.

(2) $S$ has a unique multiplicative right invariant mean.

(3) $S$ has a unique two-sided multiplicative invariant mean.

(4) $\beta S$ has a unique right under $\odot_\ell$ as well as $\odot_r$.

(5) $\beta S$ has a unique left zero under $\odot_\ell$ as well as $\odot_r$.

(6) $\beta S$ has a zero under $\odot_\ell$ as well as $\odot_r$.

(7) The collection of right thick subsets of $S$ is an ultrafilter.

(8) The collection of left thick subsets of $S$ is an ultrafilter.

(9) Given $f \varepsilon m(S)$ there is a unique constant function in $k_f$ where $k_f$ is as in lemma 2.11.

(10) Given $f \varepsilon m(S)$ there is a unique constant function in $L_f$ where $L_f$ is the weak* closure of the set $\{\ell_a f \mid a \varepsilon S\}$.

The proof is an easy application of theorems 2.11 and 2.12. Theorems 2.11 and 2.13 shows the relationship between uniqueness of multiplicative left invariant means and the collection of all left thick subsets forming an ultrafilter. In fact, the collection of all left thick subsets of a semigroup need not even form a filter as is the case of the semigroup strictly positive integers under usual addition or the semigroups in example 2.7. A nice characterization of semigroups for which the collection of all left thick subsets of $S$ is a filter is not known. However, we have a partial result.

<u>Theorem 2.14</u>: Let $S$ be an extremely left amenable semigroup. Then the collection Ⓛ of all left thick subsets of $S$ is a filter if and only if $A \cap B \neq \phi$ for all left thick subsets $A, B$ of $S$.

Proof: The "only if" part is clear. So we have to only show that if $C \cap D \neq \phi$ for all $C, D \in (L)$ then $C \cap D \in (L)$. Let $D' = S/D$. Now $C = (C \cap D) \cup (C \cap D')$ and $C$ is left thick. So theorem 1.12 gives that either $C \cap D$ or $(C \cap D')$ is left thick. However $D$ is left thick and any two left thick subsets have non-empty intersection. So $C \cap D'$ cannot be left thick. So $C \cap D \in (L)$. Thus the theorem.

We are able to get that the collection of left thick subsets is a filter for another class of semigroups than the ones given in theorem 2.14. First we give a definition.

Definition 2.15: Let $S$ be a semigroup. A left thick subset $M$ of $S$ is called the smallest left thick subset if $A \supset M$ whenever $A$ is a left thick subset of $S$. A left thick subset $B$ of $M$ is called a minimal left thick subset of $S$ if $B$ contains no left thick subset of $S$ other than itself.

Theorem 2.17: Let $S$ be a semigroup and let the collection $(L)$ of all left thick subsets of $S$ form a filter. Let there be a finite left thick subset of $S$. Then there is a minimal left thick subset $K$ of $S$ so that $K$ is a right ideal of $S$. Further, such a right ideal $K$ can be expressed as a direct product $G \times F$ where $G$ is a finite group and $F$ is a finite set with left zero multiplication. (That is $xy=x$ for all $x, y \in F$).

Proof: Let $T$ be a finite subset which belongs to $(L)$. Let $(H)$ be the collection of all subsets of $T$ which are left thick in $S$. Since $(H)$ is finite and $(L)$ is a filter we have that the intersection $K$ of all members in Ⓗ is again in Ⓛ and hence left thick. Clearly $K$ is a minimal left thick subset of $S$. To see that $K$ is a right ideal notice that if $a \varepsilon S$ then $Ka$ is also left thick since $K$ is. So $K \cap Ka$ is left thick because Ⓛ is a filter. So $K = K \cap Ka$ since $K$ is minimal left thick. So $Ka$ $K$. However $|Ka| \leq |K|$ and $|Ka|$ is finite. So $Ka = K$. So we get at the same time that $K$ is a right ideal and also that $|Ka| = |K|$ for all $a \varepsilon S$. So if $x, y \varepsilon K$ and $a \varepsilon S$ and $xa = ya$ then $x = y$. So $K$ is right cancellative. Then $K$ cannot have a proper left ideal of itself. By a theorem in [11] we have that $K$ can be expressed as $G \times F$ where $G$ is a group and $F$ is the left zero semigroup.

<u>Theorem 2.18</u>: Let $S$ be a right cancellative semigroup and Ⓛ the collection of all left thick subsets of $S$. Let there be a smallest left thick subset $A$ of $S$. Then Ⓛ is a filter and $A$ can be expressed as a direct product $G \times E$ where $G$ is a group and $E$ is a semigroup with left zero multiplication.

We omit the proof since it is similar to that of theorem 2.17.

<u>PROBLEMS</u>:

(1) Let $S$ be a semigroup with a unique multiplicative left invariant mean. Is the right multiplicative invariant mean on $S$ always unique?.

(2) What are all the cardinal numbers $\alpha$ so that there is a semigroup $S$ so that the set of all multiplicative invariant means on $S$ has cardinality $\alpha$?.

(3) What are all the ordered pairs $(\alpha,\beta)$ so that there is a semigroup $S$ whose set of all left invariant means has cardinality $\alpha$ and the set of all right - invariant means has cardinality $\beta$?.

(4) Find the algebraic structure of R-semigroups.

(5) Find the algebraic structure of V-semigroups.

(6) Find good necessary and sufficient conditions on a semigroup $S$ so that the set of all its left thick subsets forms a filter.

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