Universidad de Los Andes
Facultad de Ciencias
Departamento de Matemáticas

Existence, Stability and Smoothness of a Bounded Solution for Nonlinear Time-Varying Thermoelastic Plate Equations

H. LEIVA AND Z. SIVOLI

Notas de Matemática

Serie: Pre-Print

No. 233

Mérida - Venezuela

2004

Existence, Stability and Smoothness of a Bounded Solution for Nonlinear Time-Varying Thermoelastic Plate Equations

H. LEIVA AND Z. SIVOLI

Abstract

In this paper we study the existence, stability and the smoothness of a bounded solution of the following nonlinear time-varying thermoelastic plate Equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_{tt} + \Delta^2 u + \alpha\Delta\theta = f_1(t,u,\theta) \quad t \geq 0, \quad x \in \Omega, \\ \theta_t - \beta\Delta\theta - \alpha\Delta u_t = f_2(t,u,\theta), \quad t \geq 0, \quad x \in \Omega, \\ \theta = u = \Delta u = 0, \quad t \geq 0, \quad x \in \partial\Omega, \end{cases}$$

where $\alpha \neq 0$, $\beta > 0$, Ω is a sufficiently regular bounded domain in $\mathbb{R}^N (N \geq 1)$ and $f_1^e, f_2^e : \mathbb{R} \times L^2(\Omega)^2 \to L^2(\Omega)$ define by $f^e(t,u,\theta)(x) = f(t,u(x),\theta(x)), \quad x \in \Omega$ are continuous and locally Lipschitz functions. First, we prove that the linear system ($f_1 = f_2 = 0$) generates an analyitic strongly continuous semigroups which decays exponentially to zero. Second, under some additional condition we prove that the non-linear system has a bounded solution which is exponentially stable, and for a large class of functions f_1, f_2 this bounded solution is almost periodic. Finally, we use the analyticity of the semigroup generated by the linear system to prove the smoothness of the bounded solution.

Key words. thermoelastic plate equation, bounded solutions, exponential stability, smoothness.

AMS(MOS) subject classifications. primary: 34G10; secondary: 35B40.

1 Introduction

In this paper we study the existence, stability and the smoothness of a bounded solution of the following nonlinear time-varying thermoelastic plate Equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_{tt} + \Delta^2 u + \alpha\Delta\theta = f_1(t,u,\theta) & t \geq 0, \quad x \in \Omega, \\ \theta_t - \beta\Delta\theta - \alpha\Delta u_t = f_2(t,u,\theta), & t \geq 0, \quad x \in \Omega, \\ \theta = u = \Delta u = 0, \quad t \geq 0, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $\alpha \neq 0$, $\beta > 0$, Ω is a sufficiently regular bounded domain in $\mathbb{R}^N (N \geq 1)$and u, θ denote the vertical deflection and the temperature of the plate respectively.

We shall assume the following hypothesis:

H1) $f_1^e, f_2^e : \mathbb{R} \times L^2(\Omega)^2 \to L^2(\Omega)$ define by $f^e(t,u,\theta)(x) = f(t,u(x),\theta(x)), \;\; x \in \Omega$ are continuous and locally Lipschitz functions. i.e., for every ball B_ρ in $L^2(\Omega)^2$ of radius $\rho > 0$ there exist constants $L_1(\rho), L_2(\rho) > 0$ such that for all $(u,\theta),(v,\eta) \in B_\rho$

$$\|f_i^e(t,u,\theta) - f_i^e(t,v,\eta)\|_{L^2} \leq L_i(\rho)\{\|u-v\|_{L^2} + \|\theta-\eta\|_{L^2}\}, \quad t \in \mathbb{R}. \tag{2}$$

H2) there exists $L_f > 0$ such that

$$\|f_i(t,0,0)\| \leq L_f, \quad \forall t \in \mathbb{R}, \quad i = 1,2. \tag{3}$$

Observation 1.1 *The hypothesis H1) can be satisfied in the case that $f_1, f_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and globally Lipschitz functions with Lipschitz constants $L_1, L_2 > 0$. i.e.,*

$$|f_i(t,u,\theta) - f_i(t,v,\eta)| \leq L_i\{|u-v|^2 + |\theta-\eta|^2\}, \quad t,u,v,\theta,\eta \in \mathbb{R}, i = 1,2. \tag{4}$$

The derivation of the unperturbed ($f_i = 0, i = 1,2$) thermoelastic plate equation

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0, & t \geq 0, \quad x \in \Omega, \\ \theta_t - \beta\Delta\theta - \alpha\Delta w_t = 0, & t \geq 0, \quad x \in \Omega, \\ \theta = w = \Delta w = 0, & t \geq 0, \quad x \in \partial\Omega, \end{cases} \tag{5}$$

can be found in J. Lagnese [13], where the author discussed stability of various plate models. J.U. Kim [11](1992) studied the system (5) with the following homogeneous Dirichlet boundary condition

$$\theta = \frac{\partial w}{\partial \eta} = w = 0, \quad \text{on} \;\; \partial\Omega,$$

and he proved the exponential decay of the energy. Also, **linear** thermoelastic plate equations has been studied in [22], [3], [4], [5], [15], [16] and [23] which conform a good reference.

One point that makes this work different from others authors works, is that here we study the existence and stability of a bounded solution for the **non-linear** thermoelastic plate equation (1). First, we prove that the linear system ($f_1 = f_2 = 0$) generates an analyitic strongly continuous semigroups which decays exponentially to zero. Second, under some additional condition we prove that the non-linear system has a bounded solution which is exponentially stable, and for a large class of functions f_1, f_2 this bounded solution is almost periodic. Finally, we use the analyticity of the semigroup generated by the linear system to prove the smoothness of the bounded solution. Some notation for this work can be found in [17], [18], [19], [20] and [1].

2 Abstract Formulation of the Problem

In this section we choose the space in which this problem will be set as an abstract ordinary differential equation.

Let $X = L^2(\Omega) = L^2(\Omega, \mathbb{R})$ and consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega, \mathbb{R}) \cap H_0^1(\Omega, \mathbb{R}). \tag{6}$$

The operator A has the following very well known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$

each one with finite multiplicity γ_n equal to the dimension of the corresponding eigenspace. Therefore,

a) there exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvectors of A.

b) for all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < x, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n x, \tag{7}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_n x = \sum_{k=1}^{\gamma_n} < x, \phi_{n,k} > \phi_{n,k}. \tag{8}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in X and $x = \sum_{n=1}^{\infty} E_n x, \quad x \in X$.

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x. \tag{9}$$

d) The fractional powered spaces X^r are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \tag{10}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert Space with norm and inner product given by:

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|u\|_r^2 + \|v\|^2, \quad < w, v >_r = < A^r w, A^r v > + < w, v > .$$

Hence, the equation (1) can be written as an abstract system of ordinary differential equation in $Z_1 = X^1 \times X \times X$ as follows

$$\begin{cases} u' = v \\ v' = -A^2 u + \alpha A\theta + f_1(t, u, \theta) \\ \theta' = -\beta A\theta - \alpha A v + f_2(t, u, \theta). \end{cases} \tag{11}$$

Finally, the system can be written as first order system of ordinary differential equations in the Hilbert space $Z_1 = X^1 \times X \times X$ as follows:

$$z' = \mathcal{A} z + F(t, z) \quad z \in Z_1, \quad t \geq 0, \tag{12}$$

where $F : \mathbb{R} \times Z_1 \to Z_1$,

$$z = \begin{bmatrix} u \\ v \\ \theta \end{bmatrix}, \quad F(t, u, v, \theta) = \begin{bmatrix} 0 \\ f_1^e(t, u, \theta) \\ f_2^e(t, u, \theta) \end{bmatrix}$$

and

$$\mathcal{A} = \begin{bmatrix} 0 & I_X & 0 \\ -A^2 & 0 & \alpha A \\ 0 & -\alpha A & -\beta A \end{bmatrix}, \tag{13}$$

is an unbounded linear operator with domain

$$D(\mathcal{A}) = \{u \in H^4(\Omega) : u = \Delta u = 0\} \times D(A) \times D(A).$$

>From the hypothesis H1) we get that F is locally Lipschitz functions. i.e., for every ball B_ρ in Z_1 of radius $\rho > 0$ there exists constant L_ρ such that

$$\|F(t,z) - F(t,y)\| \leq L_\rho \|z - y\|, \quad t \in \mathbb{R}, z, y \in Z_1, \tag{14}$$

and from the hypothesis H2) we obtain the following estimate

$$\|F(t,0)\| \leq L_F = \sqrt{2\mu(\Omega)} L_f, \quad t \in \mathbb{R}, \tag{15}$$

wher $\mu(\Omega)$ is the lebesgue measure of Ω.

3 The Linear Thermoelastic Plate Equation

In this section we shall prove that the linear unbounded operator $\mathcal{A}$ given by the linear thermoelastic plate equation (5) generates an analytic strongly continuous semigroup which decays exponentially to zero. To this end, we will use the following Lemma from [21].

Lemma 3.1 *Let Z be a separable Hilbert space and $\{A_n\}_{n\geq 1}$, $\{P_n\}_{n\geq 1}$ two families of bounded linear operators in Z with $\{P_n\}_{n\geq 1}$ being a complete family of orthogonal projections such that*

$$A_n P_n = P_n A_n, \quad n = 1, 2, 3, \ldots \tag{16}$$

Define the following family of linear operators

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \qquad t \geq 0. \tag{17}$$

Then:

(a) $T(t)$ is a linear bounded operator if

$$\|e^{A_n t}\| \leq g(t), \quad n = 1, 2, 3, \dots \tag{18}$$

for some continuous real-valued function $g(t)$.

(b) under the condition (18) $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup in the Hilbert space Z whose infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \qquad z \in D(\mathcal{A}) \tag{19}$$

with

$$D(\mathcal{A}) = \{z \in Z : \sum_{n=1}^{\infty} \|A_n P_n z\|^2 < \infty\} \tag{20}$$

(c) the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is given by

$$\sigma(\mathcal{A}) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{A}_n)}, \tag{21}$$

where $\bar{A}_n = A_n P_n$.

Theorem 3.1

The operator $\mathcal{A}$ given by (13), is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z_1, \quad t \geq 0 \tag{22}$$

where $\{P_j\}_{j\geq 0}$ is a complete family of orthogonal projections in the Hilbert space Z_1 given by

$$P_j = \begin{bmatrix} E_j & 0 & 0 \\ 0 & E_j & 0 \\ 0 & 0 & E_j \end{bmatrix}, \quad , j = 1, 2, \dots, \infty, \tag{23}$$

and

$$A_j = B_j P_j, \quad B_j = \begin{bmatrix} 0 & 1 & 0 \\ -\lambda_j^2 & 0 & \alpha\lambda_j \\ 0 & -\alpha\lambda_j & -\beta\lambda_j. \end{bmatrix}, \ j \geq 1 \tag{24}$$

Moreover, the eigenvalues $\sigma_1(j)$, $\sigma_2(j)$, $\sigma_3(j)$ of the matrix B_j are simple and given by:

$$\sigma_1(j) = -\lambda_j \rho_1, \quad \sigma_2(j) = -\lambda_j \rho_2, \quad \sigma_3(j) = -\lambda_j \rho_3$$

where $\rho_i > 0, i = 1, 2, 3$ are the roots of the characteristic equation

$$\rho^3 - \beta\rho^2 + (1+\alpha^2)\rho - \beta = 0,$$

and this semigroup decays exponentially to zero

$$\|T(t)\| \leq Me^{-\mu t}, \quad t \geq 0, \tag{25}$$

where

$$\mu = \lambda_1 min\{Re(\rho) : \quad \rho^3 - \beta\rho^2 + (1+\alpha^2)\rho - \beta = 0\}$$

Proof. Let us compute $\mathcal{A}z$:

$$\begin{aligned}
\mathcal{A}z &= \begin{bmatrix} 0 & I & 0 \\ -A^2 & 0 & \alpha A \\ 0 & -\alpha A & -\beta A \end{bmatrix} \begin{bmatrix} w \\ v \\ \theta \end{bmatrix} \\
&= \begin{bmatrix} v \\ -A^2 w + \alpha A\theta \\ -\alpha A v - \beta A\theta \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^{\infty} E_j v \\ -\sum_{j=1}^{\infty} \lambda_j^2 E_j w + \alpha \sum_{j=1}^{\infty} \lambda_j E_j \theta \\ -\alpha \sum_{j=1}^{\infty} \lambda_j E_j v - \beta \sum_{j=1}^{\infty} \lambda_j E_j \theta \end{bmatrix} \\
&= \sum_{j=1}^{\infty} \begin{bmatrix} E_j v \\ -\lambda_j^2 E_j w + \alpha\lambda_j E_j\theta \\ -\alpha\lambda_j E_j v - \beta\lambda_j E_j\theta \end{bmatrix} \\
&= \sum_{j=1}^{\infty} \begin{bmatrix} 0 & 1 & 0 \\ -\lambda_j^2 & 0 & \alpha\lambda_j \\ 0 & -\alpha\lambda_j & -\beta\lambda_j \end{bmatrix} \begin{bmatrix} E_j & 0 & 0 \\ 0 & E_j & 0 \\ 0 & 0 & E_j \end{bmatrix} \begin{bmatrix} w \\ v \\ \theta \end{bmatrix} \\
&= \sum_{j=1}^{\infty} A_j P_j z.
\end{aligned}$$

It is clear that $A_j P_j = P_j A_j$. Now, we need to check condition (18) from Lemma 3.1. To this end, we have to compute the spectrum of the matrix B_j. The characteristic equation of B_j is given by

$$\lambda^3 + \beta\lambda_j\lambda^2 + \lambda_j^2(1+\alpha^2)\lambda + \beta\lambda_j^3 = 0.$$

Then,

$$\left(\frac{\lambda}{\lambda_j}\right)^3 + \beta\left(\frac{\lambda}{\lambda_j}\right)^2 + \lambda_j^2(1+\alpha^2)\left(\frac{\lambda}{\lambda_j}\right) + \beta = 0.$$

Letting $\frac{\lambda}{\lambda_j} = -\rho$ we obtain the equation

$$\rho^3 - \beta\rho^2 + (1+\alpha^2)\rho - \beta = 0. \tag{26}$$

>From Routh Hurwitz Theorem we obtain that the real part of the roots ρ_1, ρ_2, ρ_3 of equation (26) are positive. Therefore, the eigenvalues $\sigma_1(j)$, $\sigma_2(j)$, $\sigma_3(j)$ of B_j are given by

$$\sigma_1(j) = -\lambda_j\rho_1, \quad \sigma_2(j) = -\lambda_j\rho_2, \quad \sigma_3(j) = -\lambda_j\rho_3, \tag{27}$$

Since the eigenvalues of B_j are simple, there exists a complete family of complementaries projections $\{q_i(j)\}_{i=1}^{3}$ in $\mathbb{R}^3$ such that

$$\begin{cases} B_j & = \sigma_1(j)q_1(j) + \sigma_1(j)q_2(j) + \sigma_1(j)q_3(j) \\ e^{B_j t} & = e^{-\lambda_j\rho_1 t}q_1(j) + e^{-\lambda_j\rho_2 t}q_2(j) + e^{-\lambda_j\rho_3 t}q_3(j), \end{cases}$$

where $q_i(j), \quad i = 1,2,3$ are given by:

$$\begin{aligned}
q_1(j) &= \frac{1}{(\rho_1-\rho_2)(\rho_1-\rho_3)} \begin{bmatrix} \rho_2\rho_3 - 1 & \frac{\rho_2+\rho_3}{\lambda_j} & \frac{\alpha}{\lambda_j} \\ \lambda_j(\rho_3-\rho_2) & \rho_2\rho_3 - 1 - \alpha^2 & \alpha(\rho_2+\rho_3-\beta) \\ \lambda_j\alpha & -\alpha(\rho_2+\rho_3-\beta) & (\rho_3-\beta)^2 - \alpha^2, \end{bmatrix} \\
q_2(j) &= \frac{1}{(\rho_2-\rho_1)(\rho_2-\rho_3)} \begin{bmatrix} \rho_1\rho_3 - 1 & \frac{\rho_1+\rho_3}{\lambda_j} & \frac{\alpha}{\lambda_j} \\ \lambda_j(\rho_3-\rho_1) & \rho_1\rho_3 - 1 - \alpha^2 & \alpha(\rho_1+\rho_3-\beta) \\ \lambda_j\alpha & -\alpha(\rho_1+\rho_3-\beta) & (\rho_3-\beta)^2 - \alpha^2, \end{bmatrix} \\
q_3(j) &= \frac{1}{(\rho_3-\rho_1)(\rho_3-\rho_2)} \begin{bmatrix} \rho_1\rho_2 - 1 & \frac{\rho_1+\rho_2}{\lambda_j} & \frac{\alpha}{\lambda_j} \\ \lambda_j(\rho_2-\rho_1) & \rho_1\rho_2 - 1 - \alpha^2 & \alpha(\rho_1+\rho_2-\beta) \\ \lambda_j\alpha & -\alpha(\rho_1+\rho_2-\beta) & (\rho_2-\beta)^2 - \alpha^2. \end{bmatrix}
\end{aligned}$$

Therefore,

$$\begin{cases} A_j & = \sigma_1(j)P_{j1} + \sigma_1(j)P_{j2} + \sigma_1(j)P_{j3} \\ e^{A_j t} & = e^{-\lambda_j\rho_1 t}P_{j1} + e^{-\lambda_j\rho_2 t}P_{j2} + e^{-\lambda_j\rho_3 t}P_{j3}, \end{cases}$$

and

$$\mathcal{A}z = \sum_{j=1}^{\infty} \{\sigma_1(j)P_{j1}z + \sigma_2(j)P_{j2}z + \sigma_3(j)P_{j3}z\}, \tag{28}$$

where, $P_{ji} = q_i(j)P_j$ is a complete family of orthogonal projections in Z_1.

To prove that $e^{A_n t}P_n : Z_1 \to Z_1$ satisfies condition (18) from Lemma 3.1, it will be enough to prove for example that $e^{-\lambda_n\rho_2 t}q_2(n)P_n, n = 1,2,3,\ldots$ satisfies the condition. In fact, consider

$z = (z_1, z_2, z_3)^T \in Z_1$ such that $\|z\| = 1$. Then,

$$\|z_1\|_1^2 = \sum_{j=1}^{\infty} \lambda_j^2 \|E_j z_1\|^2 \leq 1, \quad \|z_2\|_X^2 = \sum_{j=1}^{\infty} \|E_j z_2\|^2 \leq 1 \text{ and } \|z_3\|_X^2 = \sum_{j=1}^{\infty} \|E_j z_3\|^2 \leq 1.$$

Therefore, $\lambda_j \|E_j z_1\| \leq 1, \quad \|E_j z_2\| \leq 1, \quad \|E_j z_3\| \leq 1 \;\; j = 1, 2, \ldots$. Then,

$$|e^{-\lambda_j \rho_2 t} q_2(n) P_n z\|_{Z_1}^2 =$$

$$\frac{e^{-2\lambda\rho_2 t}}{(\rho_2-\rho_1)^2(\rho_2-\rho_3)^2} \left\| \begin{array}{c} (\rho_1\rho_3 - 1)E_n z_1 + \frac{\rho_1+\rho_3}{\lambda_n} E_n z_2 + \frac{\alpha}{\lambda_n} E_n z_3 \\ \lambda_n(\rho_3-\rho_1)E_n z_1 + (\rho_1\rho_3 - 1 - \alpha^2)E_n z_2 + \alpha(\rho_1+\rho_3-\beta)E_n z_3 \\ \lambda_n \alpha E_n z_1 + -\alpha(\rho_1+\rho_3-\beta)E_n z_2 + [(\rho_3-\beta)^2 - \alpha^2]E_n z_3 \end{array} \right\|_{Z_1}^2$$

$$\begin{aligned}
&= e^{-2\lambda_n \rho_2 t} \sum_{j=1}^{\infty} \lambda_j^2 \|E_j \left((\rho_1\rho_3 - 1)E_n z_1 + \frac{\rho_1+\rho_3}{\lambda_j} E_n z_2 + \frac{\alpha}{\lambda_j} E_n z_3 \right) \|^2 \\
&+ e^{-2\lambda_n \rho_2} \sum_{j=1}^{\infty} \|E_j \left(\lambda_n(\rho_3-\rho_1)E_n z_1 + (\rho_1\rho_3 - 1 - \alpha^2)E z_2 + \alpha(\rho_1+\rho_3-\beta)E n z_3 \right) \|^2 \\
&+ e^{-2\lambda_n \rho_2 t} \sum_{j=1}^{\infty} \|E_j \left(\lambda_n \alpha E_n z_1 + -\alpha(\rho_1+\rho_3-\beta)E_n z_2 + [(\rho_3-\beta)^2 - \alpha^2]E_n z_3 \right) \|^2 \\
&= e^{-2\lambda_n \rho_2 t} \lambda_n^2 \|(\rho_1\rho_3 - 1)E_n z_1 + \frac{\rho_1+\rho_3}{\lambda_n} E_n z_2 + \frac{\alpha}{\lambda_n} E_n z_3\|^2 \\
&+ e^{-2\lambda_n \rho_2 t} \|\lambda_n(\rho_3-\rho_1)E_n z_1 + (\rho_1\rho_3 - 1 - \alpha^2)E_n z_2 \alpha(\rho_1+\rho_3-\beta)E_n z_3\|^2 \\
&+ e^{-2\lambda_n \rho_2 t} \|\lambda_\alpha E_n z_1 + -\alpha(\rho_1+\rho_3-\beta)E_n z_2 + [(\rho_3-\beta)^2 - \alpha^2]E_n z_3\|^2 \\
&\leq e^{-2\lambda_n \rho_2 t} \left[|\rho_1\rho_3 - 1| + \rho_1 + \rho_3 + \alpha\right]^2 \\
&+ e^{-2\lambda_n \rho_2 t} \left[|\rho_3 - \rho_1| + |\rho_1\rho_3 - 1 - \alpha^2| + \alpha|\rho_1+\rho_3-\beta|\right]^2 \\
&+ e^{-2\lambda_n \rho_2 t} \left[\alpha + \alpha|\rho_1+\rho_3-\beta| + |(\rho_3-\beta)^2 - \alpha^2|\right]^2 \\
&\leq M^2 e^{-2\lambda_n \rho_2 t}.
\end{aligned}$$

where $M = M(\alpha, \beta) \geq 1$ depending on α and β. Then we have,

$$\|e^{-\lambda_n \rho_2 t} q_2(n) P_n\|_{Z_1} \leq M(\alpha, \beta) e^{-\lambda_n \rho_2 t}, \quad t \geq 0 \;\; n = 1, 2, \ldots.$$

In the same way e obtain that

$$\begin{aligned}\|e^{-\lambda_n\rho_1 t}q_1(n)P_n\|_{Z_1} &\leq M(\alpha,\beta)e^{-\lambda_n\rho_1 t}, \quad t\geq 0 \quad n=1,2,\ldots,\\ \|e^{-\lambda_j\rho_3 t}q_3(n)P_n\|_{Z_1} &\leq (\alpha,\beta)e^{-\lambda_n\rho_3 t}, \quad t\geq 0 \quad n=1,2,\ldots.\end{aligned}$$

Therefore,

$$\|e^{A_n t}P_n\|_{Z_1}\leq M(\alpha,\beta)e^{-\mu t}, \quad t\geq 0 \quad n=1,2,\ldots,$$

were

$$\mu=\lambda_1\min\{\mathrm{Re}(\rho): \quad \rho^3-\beta\rho^2+(1+\alpha^2)\rho-\beta=0\}.$$

Hene, applying Lemma 3.1 we obtain that $\mathcal{A}$ generates a strongly contnuous semigroup given by (22). Next, we prove this semigroup decays exponentially to zero. In fact,

$$\begin{aligned}\|T(t)z\|^2 &= \sum_{j=1}^{\infty}\|e^{A_j t}P_j z\|^2\\ &\leq \sum_{j=1}^{\infty}\|e^{A_j t}\|^2\|P_j z\|^2\\ &\leq M^2(\alpha,\beta)e^{-2\mu t}\sum_{j=1}^{\infty}\|P_j z\|^2\\ &= M^2(\alpha,\beta)e^{-2\mu}\|z\|^2.\end{aligned}$$

Therefore,

$$\|T(t)\|\leq M(\alpha,\beta)e^{-\mu t}, \quad t\geq 0.$$

To prove the analyticity of $\{T(t)\}_{t\geq 0}$, we shall use Theorem 1.3.4 from [10]. To this end, it will be enough to prove that the operator $-\mathcal{A}$ is sectorial. In order to construct the sector we shall consider the following 3×3 matrices

$$\overline{K}_n=\begin{bmatrix}1 & 1 & 1\\ \lambda_n\rho_1 & \lambda_n\rho_2 & \lambda_n\rho_3\\ \frac{\alpha\rho_1}{\rho_1-\beta}\lambda_n & \frac{\alpha\rho_2}{\rho_2-\beta}\lambda_n & \frac{\alpha\rho_3}{\rho_3-\beta}\lambda_n\end{bmatrix}, \tag{29}$$

$$\overline{K}_n^{-1}=\frac{1}{a(\alpha,\beta)\lambda_n}\begin{bmatrix}a_{11} & -a_{12} & a_{13}\\ -a_{21} & a_{22} & -a_{23}\\ a_{31} & -a_{32} & a_{33}\end{bmatrix}, \tag{30}$$

where

$$\begin{aligned}
a_{11} &= \frac{\alpha\rho_3\rho_2(\rho_2-\rho_3)}{(\rho_3-\beta)(\rho_2-\beta)}, \quad a_{12} = \frac{\alpha\rho_3\rho_1(\rho_1-\rho_3)}{(\rho_3-\beta)(\rho_1-\beta)}, \quad a_{13} = \frac{\alpha\rho_2\rho_1(\rho_1-\rho_2)}{(\rho_2-\beta)(\rho_1-\beta)},\\
a_{21} &= \frac{\alpha\beta(\rho_2-\rho_3)}{(\rho_3-\beta)(\rho_2-\beta)}, \quad a_{22} = \frac{\alpha\beta(\rho_1-\rho_3)}{(\rho_3-\beta)(\rho_1-\beta)}, \quad a_{23} = \frac{\alpha\beta(\rho_1-\rho_2)}{(\rho_2-\beta)(\rho_1-\beta)},\\
a_{31} &= (\rho_3-\rho_2), \quad a_{32} = (\rho_3-\rho_1), \quad a_{33} = (\rho_2-\rho_1),\\
a(\alpha,\beta) &= \frac{\alpha\rho_3\rho_2}{(\rho_3-\beta)} + \frac{\alpha\rho_1\rho_3}{(\rho_1-\beta)} + \frac{\alpha\rho_2\rho_1}{(\rho_2-\beta)} - \frac{\alpha\rho_1\rho_2}{(\rho_1-\beta)} - \frac{\alpha\rho_3\rho_1}{(\rho_3-\beta)} - \frac{\alpha\rho_2\rho_3}{(\rho_2-\beta)}.
\end{aligned}$$

Then,

$$B_n = \overline{K}_n^{-1}\overline{J}_n\overline{K}_n, \quad n = 1, 2, 3, \ldots, \tag{31}$$

with

$$\overline{J}_n = \begin{bmatrix} -\lambda_n\rho_1 & 0 & \\ 0 & -\lambda_n\rho_2 & \\ 0 & 0 & -\lambda_n\rho_3 \end{bmatrix}.$$

Next, we define the following two linear bounded operators

$$K_n : X \times X \times X \to X^1 \times X \times X, \quad K_n^{-1} : X^1 \times X \times X \to X \times X \times X, \tag{32}$$

as follows $K_n = \overline{K}_n^{-1}P_n$ and $K_n = \overline{K}_n^{-1}P_n$. Now we will obtain bounds for $\|K_n^{-1}\|$ and $\|_n\|$. Consider $z = (z_1, z_2, z_3)^T \in Z_1 = X^1 \times X \times X$, such that $\|z\|_{Z_1} = 1$. Then,

$$\|z_1\|_1^2 = \sum_{j=1}^{\infty} \lambda_j^2\|E_jz_1\|^2 \leq 1, \quad \|z_2\|_X^2 = \sum_{j=1}^{\infty}\|E_jz_2\|^2 \leq 1 \text{ and } \|z_3\|_X^2 = \sum_{j=1}^{\infty}\|E_jz_3\|^2 \leq 1.$$

Therefore, $\lambda_j\|E_jz_1\| \leq 1$, $\|E_jz_2\| \leq 1$, $\|E_jz_3\| \leq 1$, $j = 1, 2, \ldots$. Then,

$$\begin{aligned}
\|K_n^{-1}z\|_{X\times X\times X}^2 &= \frac{1}{a(\alpha,\beta)\lambda_n^2}\left\|\begin{bmatrix} a_{11}E_nz_1 - a_{12}E_nz_2 + a_{13}E_nz_3 \\ -a_{21}E_nz_1 + a_{22}E_nz_2 - a_{23}E_nz_3 \\ a_{31}E_nz_1 - a_{32}E_nz_2 + a_{33}E_nz_3 \end{bmatrix}\right\|_{X\times X}^2\\
&= \frac{1}{a(\alpha,\beta)\lambda_n^2}\|a_{11}E_nz_1 - a_{12}E_nz_2 + a_{13}E_nz_3\|^2\\
&+ \frac{1}{a(\alpha,\beta)\lambda_n^2}\|-a_{21}E_nz_1 + a_{22}E_nz_2 - a_{23}E_nz_3\|^2\\
&+ \frac{1}{a(\alpha,\beta)\lambda_n^2}\|a_{31}E_nz_1 - a_{32}E_nz_2 + a_{33}E_nz_3\|^2\\
&\leq \frac{\Gamma_1^2(\alpha,\beta)}{\lambda_n^2}.
\end{aligned}$$

Therefore,

$$\|K_n^{-1}\|_{L(X^1\times X\times X,\quad X\times X\times X)} \leq \frac{\Gamma_1(\alpha,\beta)}{\lambda_n}. \tag{33}$$

Next, we will find a bound for $\|K_n\|_{L(X\times X\times X,\quad X^1\times X\times X)}$. To this end we consider $z = (z_1, z_2, z_3)^T \in Z = X \times X \times X$, with $\|z\|_Z = 1$. Then,

$$\|z_i\|^2 = \sum_{j=1}^{\infty} \|E_j z_i\|^2 \leq 1, \quad i = 1, 2, 3.$$

Therefore, $\|E_j z_i\| \leq 1, \quad i = 1, 2, 3, \quad j = 1, 2, \ldots,$ which implies,

$$\begin{aligned}
\|K_n z\|^2_{X^\alpha\times X} &= \left\|\left[\begin{array}{c} E_n z_1 + E_n z_2 + E_n z_3 \\ \lambda_n\rho_1 E_n z_1 + \lambda_n\rho_2 E_n z_2 + \lambda_n\rho_3 E_n z_3 \\ \frac{\alpha\rho_1\lambda_n}{\rho_1-\beta} E_n z_1 + \frac{\alpha\rho_2\lambda_n}{\rho_2-\beta} E_n z_2 + \frac{\alpha\rho_3\lambda_n}{\rho_3-\beta} E_n z_3 \end{array}\right]\right\|^2_{X^1\times X\times X} \\
&= \lambda_n^2 \|E_n z_1 + E_n z_2 + E_n z_3\|^2 \\
&+ \|\lambda_n\rho_1 E_n z_1 + \lambda_n\rho_2 E_n z_2 + \lambda_n\rho_3 E_n z_3\|^2 \\
&+ \|\frac{\alpha\rho_1\lambda_n}{\rho_1-\beta} E_n z_1 + \frac{\alpha\rho_2\lambda_n}{\rho_2-\beta} E_n z_2 + \frac{\alpha\rho_3\lambda_n}{\rho_3-\beta} E_n z_3\|^2 \\
&\leq \Gamma_2^2(\alpha,\beta)\lambda_n^2.
\end{aligned}$$

Hence

$$\|K_n\|_{L(X\times X\times X, X^1\times X\times X)} \leq \Gamma_2(\alpha,\beta)\lambda_n. \tag{34}$$

Now, the matrix $\overline{J}_n$ can be written as follows

$$\begin{aligned}
-\overline{J}_n &= \operatorname{diag}\left[\lambda_n\rho_1, \lambda_n\rho_2, \lambda_n\rho_3\right] & (35) \\
&= \lambda_n\rho_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_n\rho_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_n\rho_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & (36) \\
&= \lambda_n\rho_1 q_1 + \lambda_n\rho_2 q_2 + \lambda_n\rho_1 q_1. & (37)
\end{aligned}$$

Now,define the sector S_θ as follows:

$$S_\theta = \{\lambda \in \boldsymbol{C} : \theta \leq |\arg(\lambda)| \leq \pi, \quad \lambda \neq 0\}, \tag{38}$$

where

$$\max_{i=1,2,3}\{|\arg(\rho_i)|\} < \theta < \frac{\pi}{2}.$$

If $\lambda \in S_\theta$, then λ is a value other than $\lambda_n \rho_i, i = 1, 2, 3$. Therefore,

$$\begin{aligned} \| \left(\lambda + \overline{J}_n\right)^{-1} y\|^2 &= \frac{1}{(\lambda - \lambda_n\rho_1)^2}\|q_1 y\|^2 \\ &+ \frac{1}{(\lambda - \lambda_n\rho_2)^2}\|q_2 y\|^2 \\ &+ \frac{1}{(\lambda - \lambda_n\rho_3)^2}\|q_3 y\|^2. \end{aligned}$$

Setting

$$N = \sup\left\{\frac{|\lambda|}{|\lambda - \lambda_n\rho_i|} : \lambda \in S_\theta, \quad n \geq 1; \quad i = 1, 2, 3\right\},$$

yields

$$\| \left(\lambda + \overline{J}_n\right)^{-1} y\|^2 \leq \left(\frac{N}{|\lambda|}\right)^2 \left[\|q_1 y\|^2 + \|q_2 y\|^2 + \|q_3 y\|^2\right]$$

Hence,

$$\| \left(\lambda + \overline{J}_n\right)^{-1} \| \leq \frac{N}{|\lambda|}, \quad \lambda \in S_\theta.$$

Now, if $\lambda \in S_\theta$, then

$$\begin{aligned} \mathcal{R}(\lambda, -\mathcal{A})z &= \sum_{n=1}^{\infty} (\lambda + A_n)^{-1} P_n z \\ &= \sum_{n=1}^{\infty} K_n \left(\lambda + \overline{J}_n\right)^{-1} K_n^{-1} P_n z. \end{aligned}$$

This implies,

$$\begin{aligned} \|\mathcal{R}(\lambda, \mathcal{A})z\|^2 &\leq \sum_{n=1}^{\infty} \|K_n\|^2 \|K_n^{-1}\|^2 \| \left(\lambda + \overline{J}_n\right)^{-1} \|^2 \|P_n z\|^2 \\ &\leq \left(\frac{\Gamma_1(\eta, \gamma)}{\Gamma_2(\eta, \gamma)}\right)^2 \left(\frac{N}{|\lambda|}\right)^2 \|z\|^2 \end{aligned}$$

Therefore,

$$\|\mathcal{R}(\lambda, -\mathcal{A})\| \leq \frac{R}{|\lambda|}, \quad \lambda \in S_\theta.$$

This completes the proof of the Theorem.

4 Existence of the Bounded Solution

In this section we shall prove the existence and stability of unique bounded Mild solutions of system (12).

DEFINITION 4.1 (Mild Solution) For mild solution $z(t)$ of (12) with initial condition $z(t_0) = z_0 \in Z_1$, we understand a function given by

$$z(t) = T(t - t_0)z_0 + \int_{t_0}^{t} T(t - s)F(s, z(s))ds, \quad t \in \mathbb{R}. \tag{39}$$

Observation 4.1 *It is easy to prove that any solution of (12) is a solution of (39). It may be thought that a solution of (39) is always a solution of (12) but this is not true in general. However, we shall prove in Theorem 5.2 that bounded solutions of (39) are solutions of (12).*

We shall consider $Z_b = C_b(\mathbb{R}, Z_1)$ the space of bounded and continuous functions defined in $\mathbb{R}$ taking values in Z_1. Z_b is a Banach space with supremum norm

$$\|z\|_b = \sup\{\|z(t)\|_{Z_1} : t \in \mathbb{R}\}, \quad z \in Z_b.$$

A ball of radio $\rho > 0$ and center zero in this space is given by

$$B_\rho^b = \{z \in Z_b : \|z(t)\| \leq \rho, \quad t \in \mathbb{R}\}.$$

The proof of the following Lemma is similar to Lemma 3.1 of [20].

Lemma 4.1 *Let z be in Z_b. Then, z is a mild solution of (12) if and only if z is a solution of the following integral equation*

$$z(t) = \int_{-\infty}^{t} T(t - s)F(s, z(s))ds, \quad t \in \mathbb{R}. \tag{40}$$

The following Theorem refers to bounded Mild solutions of system (12).

THEOREM 4.1

Suppose that F is Locally Lipschitz and there exists $\rho > 0$ such that

$$0 < ML_F < (\mu - ML_\rho)\rho, \tag{41}$$

where L_ρ is the Lipschitz constant of F in the ball $B^b_{2\rho}$. Then, the equation (12) has one and only one bounded mild solution $z_b(t)$ which belong B^b_ρ.

Moreover, this bounded solution is exponentially stable.

REMARK 4.1 . *For the existence of such solution, we shall prove that the following operator has a unique fixed point in the ball B^b_ρ, $\quad T : B^b_\rho \to B^b_\rho$*

$$(Tz)(t) = \int_{-\infty}^{t} T(t-s)F(s, z(s))ds, \quad t \in \mathbb{R}.$$

In fact, for $z \in B^b_\rho$ we have

$$\|Tz(t)\| \leq \int_{-\infty}^{t} Me^{-\mu(t-s)} \{L_\rho\|z(s)\| + L_F\} \leq \frac{ML_\rho\rho + ML_F}{\mu}.$$

The condition (41) implies that

$$L_\rho\rho + ML_F < \mu\rho \iff \frac{L_\rho\rho + ML_F}{\mu} < \rho.$$

Therefore, $Tz \in B^b_\rho$ for all $z \in B^b_\rho$.

Now, we shall see that T is a contraction mapping. In fact, for all $z_1, z_2 \in B^b_\rho$ we have that

$$\|Tz_1(t) - Tz_2(t)\| \leq \int_{-\infty}^{t} Me^{-\mu(t-s)} L_\rho\|z_1(s) - z_2(s)\|ds \leq \frac{ML_\rho}{\mu}\|z_1 - z_2\|_b, \quad t \in \mathbb{R}.$$

Hence,

$$\|Tz_1 - Tz_2\|_b \leq \frac{ML_\rho}{\mu}\|Tz_1 - z_2\|_b, \quad z_1, z_2 \in B^b_\rho.$$

The condition (41) implies that

$$0 < \mu - ML_\rho \iff ML_\rho < \mu \iff \frac{ML_\rho}{\mu} < 1.$$

Therefore, T has a unique fixed point z_b in B^b_ρ

$$z_b(t) = (Tz_b)(t) = \int_{-\infty}^{t} T(t-s)F(s, z_b(s))dsds, \quad t \in \mathbb{R},$$

>From Lemma 3.1, z_b is a bounded solution of the equation (39).

Now, we shall prove that $z_b(\cdot)$ is exponentially stable. To this end, we consider any other solution $z(\cdot)$ of the equation (39) such that $\|z(0) - z_b(0)\| < \frac{\rho}{2M}$. Then, $\|z(0)\| < 2\rho$. As long as $\|z(t)\|$ remains less than 2ρ we obtain the following estimate:

$$\begin{aligned} \|z(t) - z_b(t)\| &\leq \|T(t)(z(0) - z_b(0)) + \int_0^t T(t-s)\left\{F(s, z(s)) - F(s, z_b(s))\right\} ds\| \\ &\leq Me^{-\mu t}\|(z(0) - z_b(0))\| + \int_0^t Me^{-\mu(t-s)} L_\rho \|z(s) - z_b(s)\|ds. \end{aligned}$$

Then,

$$e^{\mu(t)}\|z(t) - z_b(t)\| \leq M\|(z(0) - z_b(0))\| + \int_0^t Me^{\mu s} L_\rho \|z(s) - z_b(s)\|ds.$$

Hence, applying the Gronwall's inequality we obtain

$$\|z(t) - z_b(t)\| \leq Me^{(ML_\rho - \mu)t}\|(z(0) - z_b(0))\|, \quad t \in [0, t_1)$$

>From (41) we get that $ML_\rho - \mu < 0$. Therefore $\|z(t) - z_b(t)\| \leq \rho/2$.

Hence, if $\|z(t)\| < 2\rho$ on $[0, t_1)$ with $t_1 = \inf\{t > 0 : \ \|z(t)\| < 2\rho\}$, then either $t_1 = \infty$ or $\|z(t_1)\| = 2\rho$. But, the second case contradicts the above computation, then the solution $z(t)$ remains in the ball $B^b_{2\rho}$ for all $t \geq 0$.

So,

$$\|z(t) - z_b(t)\| \leq Me^{(ML_\rho - \mu)t}\|(z(0) - z_b(0))\|, \quad t \geq 0.$$

This concludes the proof of the Theorem.

Theorem 4.2

Suppose that F globally Lipschitz with a Lipschitz constant $L > 0$ and

$$\mu > ML. \tag{42}$$

Then, the equation (12) has one and only one bounded mild solution $z_b(t)$ *on* $\mathbb{R}$.

Moreover, this bounded solution is the only bounded solution of the equation (39) and is exponentially stable in large.

Remark 4.2 . *Condition (42) implies that for* $\rho > 0$ *big enough we have the following estimate:*

$$0 < ML_F < (\mu - ML)\rho, \tag{43}$$

>From here, in a similar way we can prove that the following operator is a contraction mapping from B^b_ρ *into* B^b_ρ

$$(Tz)(t) = \int_{-\infty}^{t} T(t-s)F(s, z(s))ds, \quad t \in \mathbb{R}.$$

Therefore, T *has a unique fixed point* z_b *in* B^b_ρ

$$z_b(t) = (Tz_b)(t) = \int_{-\infty}^{t} T(t-s)F(s, z_b(s))dsds, \quad t \in \mathbb{R},$$

>From Lemma 3.1, z_b *is a bounded solution of the equation (39). Since condition (43) holds for any* $\rho > 0$ *big enough independent of* $ML < \mu$, *then* z_b *is the unique bounded solution of the equation (39).*

To prove that $z_b(t)$ *is exponentially stable in the large, we shall consider any other solution* $z(t)$ *of (39) and the following estimate*

$$\begin{aligned} \|z(t) - z_b(t)\| &\leq \|T(t)(z(0) - z_b(0)) + \int_0^t T(t-s)\left\{F(s, z(s)) - F(s, z_b(s))\right\} ds\| \\ &\leq Me^{-\mu t}\|(z(0) - z_b(0))\| + \int_0^t Me^{-\mu(t-s)} L_\rho \|z(s) - z_b(s)\| ds. \end{aligned}$$

Then,

$$e^{\mu t}\|z(t) - z_b(t)\| \leq M\|(z(0) - z_b(0))\| + \int_0^t MLe^{\mu s}\|z(s) - z_b(s)\|ds.$$

Hence, applying the Gronwall's inequality we obtain

$$\|z(t) - z_b(t)\| \leq Me^{(ML-\mu)t}\|(z(0) - z_b(0))\|, \quad t \geq 0.$$

>From (43) we know that $ML - \mu < 0$ *and therefore* $z_b(t)$ *is exponentially stable in the large*

COROLLARY 4.1 *If F is periodic in t of period τ $(F(t+\tau,\xi) = f(t,\xi)$), then the unique bounded solution given by Theorems 4.1 and 4.2 is also periodic of period τ.*

REMARK 4.3 . *Let z_b be the unique solution of (39) in the ball B^b_ρ. Then, $z(t) = z_b(t+\tau)$ is also a solution of the equation (39) lying in the ball B^b_ρ. In fact, consider $z_0 = z_b(0)$ and*

$$
\begin{aligned}
z_b(t+\tau) &= T(t+\tau)z_0 + \int_0^{t+\tau} T(t+\tau-s)F(s.z_b(s))ds \\
&= T(t)T(\tau)z_0 + \int_0^{\tau} T(t+\tau-s)F(s.z_b(s))ds \\
&+ \int_\tau^{t+\tau} T(t+\tau-s)F(s.z_b(s))ds \\
&= T(t)\left\{T(\tau)z_0 + \int_0^{\tau} T(\tau-s)F(s.z_b(s))ds\right\} \\
&+ \int_0^{t} T(t-s)F(s.z_b(s+\tau))ds \\
&= T(t)z_b(\tau) + \int_0^{t} T(t-s)F(s.z_b(s+\tau))ds.
\end{aligned}
$$

Therefore,

$$
z(t) = T(t)z_b(\tau) + \int_0^{t} T(t-s)F(s.z(s))ds,
$$

and by the uniqueness of the fixed point of the contraction mapping T in this ball, we conclude that $z_b(t) = z_b(t+\tau), \quad t \in \mathbb{R}$.

Observation 4.2 *Under some condition, the bounded solution given by Theorems 4.1 and 4.2 is almost periodic; for example we can study the case when the function F has the following form:*

$$
F(t,z) = g(z) + P(t), \quad t,\xi \in \mathbb{R}, \tag{44}
$$

where $P \in C_b(\mathbb{R}, Z_1)$ and $g : Z_1 \to Z_1$ is a locally Lipschitz function.

COROLLARY 4.2 *Suppose F has the form (44) and g is a globally Lipschitz function with a Lipschitz constant $L > 0$. Then the bounded solution $z_b(\cdot, P)$ given by Theorem 4.2 depends continuously on $P \in C_b(\mathbb{R}, Z_1)$.*

Remark 4.4 . *Let $P_1, P_2 \in C_b(\mathbb{R}, Z_1)$ and $z_b(\cdot, P_1), z_b(\cdot, P_2)$ be the bounded functions given by Theorem 4.2. Then*

$$\begin{aligned} z_b(t,\cdot,P_1) - z_b(t,\cdot,P_2) &= \int_{-\infty}^{t} T(t-s)[g(z_b(s,P_2)) - g(z_b(s,P_2))]ds \\ &+ \int_{-\infty}^{t} T(t-s)[P_1(s) - P_2(s)]ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|z_b(\cdot,P_1) - z_b(\cdot,P_2)\|_b &\leq \frac{ML}{\mu}\|z_b(\cdot,P_1) - z_b(\cdot,P_2)\|_b \\ &+ \frac{M}{\mu}\|P_1 - P_2\|_b. \end{aligned}$$

Therefore,

$$\|z_b(\cdot,P_1) - z_b(\cdot,P_2)\|_b \leq \frac{M}{\mu - ML}\|P_1 - P_2\|_b.$$

Lemma 4.2 *Suppose F is as in (44). Then, if $P(t)$ is almost periodic, then the unique bounded solution of system (39) given by Theorems 4.1 and 4.2 is also almost periodic.*

Remark 4.5 . *To prove this lemma, we shall use the following well known fact, due to S. Bohr (see J. Hale [9] in the Appendix). A function $h \in C(\mathbb{R}; Z_1)$ is almost periodic (a.p) if and only if the hull $H(h)$ of h is compact in the topology of uniform convergence.*

Here $H(h)$ is the closure of the set of translates of h under the topology of uniform convergence

$$H(h) = \overline{\{h_\tau : \tau \in \mathbb{R}\}}, \qquad h_\tau(t) = h(t+\tau), t \in \mathbb{R}.$$

Since the limit of a uniformly convergent sequence of a.p. functions is a.p., then the set A_ρ of a.p. functions in the ball B_ρ^b is closed, where ρ is given by Theorem 4.2.

Claim. *The contraction mapping T given in Theorems 4.1 and 4.2 leaves A_ρ invariant. In fact; if $z \in A_\rho$, then $h(t) = g(z(t)) + P(t)$ is also an a.p. function. Now, consider the* *function*

$$\begin{aligned}\mathcal{F}(t) = (Tz)(t) &= \int_{-\infty}^{t} T(t-s)\left\{g(z(s)) + P(s)\right\} ds \\ &= \int_{-\infty}^{t} T(t-s)h(s)ds, \quad t \in \mathbb{R}.\end{aligned}$$

Then, it is enough to establish that $H(\mathcal{F})$ is compact in the topology of uniform convergence. Let $\{\mathcal{F}_{\tau_k}\}$ be any sequence in $H(\mathcal{F})$. Since h is a.p. we can select from $\{h_{\tau_k}\}$ a Cauchy subsequence $\{h_{\tau_{k_j}}\}$, and we have that

$$\begin{aligned}\mathcal{F}_{\tau_{k_j}}(t) = \mathcal{F}(t+\tau_{k_j}) &= \int_{-\infty}^{t+\tau_{k_j}} T(t+\tau_{k_j}-s)h(s)ds \\ &= \int_{-\infty}^{t} T(t-s)h(s+\tau_{k_j})ds.\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathcal{F}_{\tau_{k_j}}(t) - \mathcal{F}_{\tau_{k_i}}(t)\| &\leq \int_{-\infty}^{t} e^{-\mu(t-s)}\|h(s+\tau_{k_j}) - h(s+\tau_{k_i})\|ds \\ &\leq \|h_{\tau_{k_j}} - h_{\tau_{k_i}}\|_b \int_{-\infty}^{t} e^{-\mu(t-s)}ds = \frac{1}{\mu}\|h_{\tau_{k_j}} - h_{\tau_{k_i}}\|_b.\end{aligned}$$

Therefore, $\{\mathcal{F}_{\tau_{k_j}}\}$ is a Cauchy sequence. So, $H(\mathcal{F})$ is compact in the topology of uniform convergence, $\mathcal{F}$ is a.p. and $TA_\rho \subset A_\rho$.

Now, the unique fixed point of T in the ball B_ρ^b lies in A_ρ. Hence, the unique bounded solution $z_b(t)$ of the equation (39) given in Theorem 4.2 is also almost periodic.

5 Smoothness of the Bounded Solution

In this section we shall prove that the bounded solution of the equation (39) given by Theorems 4.1 and 4.2 is also solution of the original equation (12). That is to say, this bounded solution is a classic solution of the equation (12). To this end, we will use the following Theorem from [12].

THEOREM 5.1

Let A on $D(A)$ be a closed operator in the Banach space X and $x \in C([a,b);X)$ with $b \le \infty$. Suppose that $x(t) \in D(A)$, $Ax(t)$ is continuous on $[a,b)$ and that the improper integrals

$$\int_a^b x(s)ds \quad and \quad \int_a^b Ax(s)ds$$

exist. Then

$$\int_a^b x(s)ds \in D(A) \quad and \quad A\int_a^b x(s)ds = \int_a^b Ax(s)ds.$$

THEOREM 5.2

The bounded Mild solution $z_b(t)$ of the equation (12) given by Theorems 4.1 and 4.2 is a classic solution of this equation on $\mathbb{R}$. i.e.,

$$z_b'(t) = \mathcal{A}z_b(t) + F(t, z_b(t)), \quad t \in \mathbb{R}.$$

REMARK 5.1 . *Let $z_b(t)$ be the only bounded mild solution of (12) given by Theorems 4.1 and 4.2. Then*

$$z(t) = \int_{-\infty}^t T(t-s)g(s)ds = \int_0^\infty T(s)g(t-s)ds, \quad t \in \mathbb{R}$$

where $g(s) = F(s, z_b(s))$. Therefore, $g \in C_b(\mathbb{R}, Z_1)$ and $\|g(s)\| \le \|g\|_b, \quad s \in (-\infty, t)$.

Let us put $x(s) = T(t-s)g(s), \quad s \in (-\infty, t)$. Then $x(s)$ is a continuous function, and since $\{T(t)\}_{t\ge 0}$ is analytic, then

$$x(s) \in D(\mathcal{A}), \quad for \quad s < t.$$

Claim. *$\mathcal{A}x(s)$ is continuous on $(-\infty, t)$ and the improper integral*

$$\int_{-\infty}^t \mathcal{A}x(s)ds, \quad t \in \mathbb{R},$$

exists.

>From Theorem 3.1, there exists a complete family of orthogonal projections $\{q_i(j)\}_{i=1}^3$ in $\mathbb{R}^3$ such that

$$\begin{cases} A_j &= \sigma_1(j)q_1(j) + \sigma_1(j)q_2(j) + \sigma_1(j)q_3(j) \\ e^{A_j t} &= e^{-\lambda_j\rho_1 t}q_1(j) + e^{-\lambda_j\rho_2 t}q_2(j) + e^{-\lambda_j\rho_3 t}q_3(j), \end{cases}$$

Hence,

$$\mathcal{A}z = \sum_{j=1}^{\infty} \{\sigma_1(j)P_{j1}z + \sigma_2(j)P_{j2}z + \sigma_3(j)P_{j3}z\} \tag{45}$$

and

$$T(t)z = \sum_{j=1}^{\infty} \left\{ e^{-\lambda_j \rho_1 t} P_{j1}z + e^{-\lambda_j \rho_2 t} P_{j2}z + e^{-\lambda_j \rho_3 t} P_{j3}z \right\},$$

where, $P_{ji} = q_i(j)P_j$ *is a complete family of orthogonal projections in* Z_1.

Therefore,

$$\mathcal{A}x(s) = \sum_{j=1}^{\infty} \left\{ -\lambda_j \rho_1 e^{-\lambda_j \rho_1 (t-s)} P_{j1}g(s) - \lambda_j \rho_2 e^{-\lambda_j \rho_2 (t-s)} p_{j2}g(s) - \lambda_j \rho_3 e^{-\lambda_j \rho_3 (t-s)} p_{j3}g(s) \right\}.$$

So,

$$\|\mathcal{A}x(s)\| \le \max_{j\ge 1} \left\{ \lambda_j |\rho_i| e^{-\lambda_j Re(\rho_i)(t-s)} : \quad i = 1,2,3. \right\} \|g\|_b.$$

Then, using the dominate convergence theorem, we get that $\mathcal{A}x(s)$ *is a continuous function on* $(-\infty, t)$. *Now, consider the following improper integrals:*

$$\begin{aligned}
\int_{-\infty}^{t} \mathcal{A}x(s)ds &= \int_0^{\infty} \mathcal{A}T(s)g(t-s)ds \\
&= \int_0^{\infty} \sum_{j=1}^{\infty} \{-\lambda_j \rho_1 e^{-\lambda_j \rho_1 s} P_{j1}g(t-s) - \lambda_j \rho_2 e^{-\lambda_j \rho_2 s} P_{j2}g(t-s) \\
&- \lambda_j \rho_3 e^{-\lambda_j \rho_3 s} P_{j3}g(t-s)\}ds \\
&= \sum_{j=1}^{\infty} \{ \int_0^{\infty} -\lambda_j \rho_1 e^{-\lambda_j \rho_1 s} P_{j1}g(t-s)ds - \int_0^{\infty} \lambda_j \rho_2 e^{-\lambda_j \rho_2 s} P_{j2}g(t-s)ds \\
&- \int_0^{\infty} \lambda_j \rho_3 e^{-\lambda_j \rho_3 s} P_{j3}g(t-s)ds\}.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\| \int_0^{\infty} -\lambda_j \rho_i e^{-\lambda_j \rho_i s} P_{ji}g(t-s)ds \| &\le \int_0^{\infty} \lambda_j |\rho_i| e^{-\lambda_j Re(\rho_i)s} \|P_{ji}g(t-s)\| ds \\
&\le \frac{|\rho_i|}{Re(\rho_i)} \|g\|_b.
\end{aligned}$$

Therefore, the improper integral

$$\int_{-\infty}^{t} \mathcal{A}x(s)ds, \quad \textit{exists.}$$

Now, from Theorem 5.1 we obtain that

$$\int_{-\infty}^{t} x(s)ds \in D(\mathcal{A}), \quad and \;\; \mathcal{A}\int_{-\infty}^{t} x(s)ds = \int_{-\infty}^{t} \mathcal{A}x(s)ds.$$

i.e.,

$$\int_{-\infty}^{t} T(t-s)g(s)ds \in D(\mathcal{A}), \quad and \;\; \mathcal{A}\int_{-\infty}^{t} T(t-s)g(s)ds = \int_{-\infty}^{t} \mathcal{A}T(t-s)g(s)ds.$$

Now, we are ready to prove that $z_b(t)$ is a solution of (12). In fact, consider

$$\begin{aligned}\frac{z_b(t+h)-z_b(t)}{h} &= \frac{1}{h}\int_{-\infty}^{t+h} T(t+h-s)g(s)ds - \frac{1}{h}\int_{-\infty}^{t} T(t-s)g(s)ds \\ &= \left(\frac{T(h)-I}{h}\right)\int_{-\infty}^{t} T(t-s)g(s)ds + \frac{1}{h}\int_{t}^{t+h} T(t+h-s)g(s)ds.\end{aligned}$$

Using the definition of infinitesimal generator of a semigroup and passing to the limit as $h \to 0^+$ we get that

$$z_b'(t) = \mathcal{A}\int_{-\infty}^{t} T(t-s)g(s)ds + T(0)g(t).$$

So,

$$z_b'(t) = \mathcal{A}z_b(t) + F(t, z_b(t)), \quad t \in \mathbb{R}.$$

References

[1] *J.M. ALONSO, J. MAWHIN AND R. ORTEGA, "Bounded solutions of second order semi-linear evolution equations and applications to the telegraph equation", J.Math. Pures Appl., 78, 49-63 (1999).*

[2] *W. ARENDT AND C.J.K. BATTY, "Almost periodic solutions fo first and second-order Cauchy problems", J.Differential Equations* **137**, *363-383 (1997).*

[3] *G. AVALOS and I. LASIECKA (1997).* Exponential Stability of a Thermoelastic System Without Mechanical Dissipation*. Rend. Instit. Mat. Univ. Trieste. Supp. Vol. XXVIII, pp 1-28.*

[4] *G. AVALOS and I. LASIECKA (2000).* Boundary Controllability of Thermoelastic Plates Via the Free Boundary Conditions*. SIAM J. Control Optim. Vol. 38, N0. 2, pp. 337-383.*

[5] *G. AVALOS and I. LASIECKA (2998).* Exponential Stability of a Thermoelastic System with Free Boundary Conditions Without Mechanical Dissipation*. SIAM J. MATH. ANAL. Vol. 29, N0. 1, pp. 155-182.*

[6] *R.F. CURTAIN and A.J. PRITCHARD, "Infinite Dimensional Linear Systems", Lecture Notes in Control and Information Sciences, Vol. 8. Springer Verlag, Berlin (1978).*

[7] *R.F. CURTAIN and H.J. ZWART, "An Introduction to Infinite Dimensional Linear Systems Theory", Texts in Applied Mathematics, Vol. 21. Springer Verlag, Berlin (1995).*

[8] *R.F. CURTAIN and A.J. PRITCHARD, "An Abstract Theory for Unbounded Control Action for Distributed Parameter Systems", SIAM. J. Control and Optimization. Vol. 15, N0 4, July 1997.*

[9] *J. HALE, "Ordinary Differential Equations", Wiley Interscience, New York (1969). Rebert E. Krieger Publ. Co., Huntington, New York*

[10] *D.HENRY, "Geometric theory of semilinear parabolic equations" , Springer, New York (1981).*

[11] *JONG UHN KIM (1992).* On the Energy Decay of a Linear Thermoelastic Bar and Plate. *SIAM J. Math Anal. Vol. 23, No. 4, pp 889.*

[12] *G.E. LADAS AND V. LAKSHMIKANTHAM, "Differential Equations in Abstract Sapces", Academic Press, Math. In science and Eng. Vol 85(1972).*

[13] *J. LAGNESE (1989).* Boundary Stabilization of Thin Plate. *SIAM Studies in Appl. Math. 10, Philadelphia.*

[14] *LUIZ A. F. de OLIVEIRA "On Reaction-Diffusion Systems" E. Journal of Differential Equations, Vol. 1998(1998), N0. 24, pp. 1-10.*

[15] *LUZ DE TERESA AND ENRIQUE ZUAZUA "Controllability of the Linear System of Thermoelastic Plates" Advances in Differential Equations, Vol. 1 (1996), pp. 369-402.*

[16] *LUZ DE TERESA "Approximate Controllability of a Semilinear Heat Equation in* $\mathbb{R}^N$ *" SIAM J. Control Optim. Vol. 36, N0. 6, pp. 2128-2147 (1998)*

[17] *H. LEIVA "Stability of a Periodic Solution for a System of Parabolic Equations" J. Applicable Analysis, Vol. 60, pp. 277-300(1996).*

[18] *H. LEIVA, "Existence of Bounded Solutions of a Second Order System with Dissipation" J. Math. Analysis and Appl.* **237***, 288-302(1999).*

[19] *L. GARCIA and H. LEIVA, "Center Manifold and Exponentially Bounded Solutions of a Forced Newtonian System with Dissipation"E. Journal Differential Equations. conf. 05, 2000, pp. 69-77.*

[20] *H. LEIVA, "Existence of Bounded Solutions of a Second Order Evolution Equation and Applications" Journal Math. Physics. Vol. 41, N0 11, 2000.*

[21] *H. LEIVA, "A Lemma on C_0-Semigroups and Applications PDEs Systems" Report Series of CDSNS 99-353.*

[22] *Y. SHIBATA (1994).* On the Exponential Decay of the Energy of a Linear Thermoelastic Plate. *Comp. Appl. Math. Vol. 13, No. 2, pp 81-102.*

[23] *J. H. ORTEGA and E. ZUAZUA (2001).* Generic Simplicity of the Spectrum and Stabilization for a Plate Equation. *SIAM J. Control Optim. Vol. 39, N0. 5, pp. 1585-1614.*