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A DIRECT PROOF OF A THEOREM ON REPRESENTABILITY OF OPERATORS

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A DIRECT PROOF OF A THEOREM ON REPRESENTABILITY OF OPERATORS [1]

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ABSTRACT. Let T be a locally compact Hausdorff space and let X be a Banach space which contains no copy of c_o. Then it is well known that every bounded linear operator $U : C_o(T) \to X$ is weakly compact and hence is representable with respect to a unique X-valued σ-additive regular Borel measure on T. The object of the present note is to provide a simple, direct and elegant proof of the representability of U and then to deduce that U is weakly compact.

Suppose T is a locally compact Hausdorff space and let $C_o(T)$ be the Banach space of all complex valued continuous functions on T vanishing at infinity, with the supremum norm $||.||_T$, given by $||f||_T = \sup_{t\in T} |f(t)|$.

Let $U : C_o(T) \to X$ be a bounded linear operator, where X is a complex Banach space containing no copy of c_o (in symbols, $c_o \not\subset X$). Then by Theorems 5.1 and 5.3 of Thomas [12] and by Theorem 5 of Bessaga and Pelczyński [2] it follows that U is weakly compact. This can also be deduced from Theorem 5 of Pelczyński [10], by considering the bounded linear operator $\hat{U} : C(\hat{T}) \to X$, where $\hat{T}$ is the Alexandroff compactification of T by adjunction of the point $\{\infty\}$, $C(\hat{T}) = \{f : \hat{T} \to \mathbf{C}, f \text{ continuous}\}$ and $\hat{U}(f) = U(f - f(\infty))$. Then by Lemma 2 of Kluvánek [9] there exists a unique

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X-valued σ-additive regular Borel measure G on T such that

$$Uf = \int_T f dG, \;\; f \in C_o(T) \qquad (*)$$

and thus the representability of U is obtained. Alternatively, following the ideas in the proof of Theorems 3.1 and 3.2 of [1], (*) can be deduced from the weak compactness of U as follows: Let $G(E) = U^{**}(\chi_E)$, $E \in \mathcal{B}(T)$. Then $U^{**}C_o(T)^{**} \subset X$, as U is weakly compact. Since G is σ-additive in $\sigma(X^{**}, X^*)$-topology,by the Orlicz-Pettis theorem G is σ-additive on $\mathcal{B}(T)$ in norm topology of X and consequently (*) holds. Moreover, by Theorem 2 of [7] and Appendix 1 of [12] it follows that G is a regular X-valued Borel measure.

In the study of representation of bounded multilinear operators on $\overset{d}{\underset{1}{\mathbf{X}}} C_o(T_i)$, where T_i, $i = 1, 2, .., d$, are locally compact Hausdorff spaces, it has been shown in Dobrakov [5] that every bounded multilinear operator $V : \overset{d}{\underset{1}{\mathbf{X}}} C_o(T_i) \to X$ admits a multilinear integral representation with respect to a unique X-valued Baire multimeasure Υ on $\overset{d}{\underset{1}{\mathbf{X}}} \mathcal{B}_o(T_i)$, whenever the complex Banach space X contains no copy of c_o. Moreover, as observed in Dobrakov and Panchapagesan [6], the example given in Pelczyński [11,p.385] serves to give a non weakly compact multilinear operator V with range in a Banach space containing no copy of c_o. Thus, for bounded multilinear operators, the concepts of weak compactness and representability are not equivalent, even though these coincide for bounded linear operators on $C_o(T)$. The latter fact is exploited above to obtain the representability of a bounded linear operator $U : C_o(T) \to X$, whenever the Banach space X contans no copy of c_o.

Because of the above situation in the case of multilinear operators, the following question arises naturally: Can the representability of U be proved without using its weak compactness? The object of the present note is to answer the question affirmatively; and it turns out that the present direct proof is quite simple and elegant.

Let $\mathcal{K}$ (resp. $\mathcal{K}_o$) be the family of all compacts (resp. compact $G_\delta s$) in T. Let $\mathcal{B}_o(T)$ be the σ-ring generated by $\mathcal{K}_o$. The members of $\mathcal{B}_o(T)$ are called the Baire sets in T. The σ-algebra of all Borel sets in T is the σ-algebra $\mathcal{B}(T)$ generated by the class of all open sets in T.

Let $\mathcal{R}$ be a ring of sets in T and let X be a complex Banach space. If $m : \mathcal{R} \to X$ is an additive set function, then m is said to be an (X-valued) vector measure. Moreover, the vector measure m is said to be σ-additive if it countably additive in the norm topology of X. The dual of X is denoted by X^* and the second dual (of X) by X^{**}.

Definition 1. *An X-valued vector measure m on $\mathcal{B}_o(T)$ (resp. $\mathcal{B}(T)$) is said to be regular, if, given $\varepsilon > 0$ and $E \in \mathcal{B}_o(T)$(resp. $E \in \mathcal{B}(T)$), there exists $C \in \mathcal{K}_o$ (resp. $C \in \mathcal{K}$)) and an open set $U \in \mathcal{B}_o(T)$ (resp. an open set U in T) such that $C \subset E \subset U$ and $||m(F)|| < \varepsilon$ for all $F \in \mathcal{B}_o(T)$ (resp. $F \in \mathcal{B}(T)$) with $F \subset U \backslash C$.*

Definition 2. *A vector measure $m : \mathcal{B}_o(T) \to X$ (resp. $\mathcal{B}(T) \to X$) is called an X-valued Baire (resp. Borel) measure on T.*

Lemma 3. *Every X-valued σ-additive Baire measure on T is regular and admits a unique X-valued σ-additive regular Borel extension on $\mathcal{B}(T)$.*

The above lemma is the same as Lemma 1 of Kluvánek [9].

Lemma 4. *Let $U : C_0(T) \to X$ be a bounded linear operator. Then there exists a weak* σ-additive vector measure G defined on $\mathcal{B}(T)$ with values in X^{**} such that*

(i) $x^*G(.)$ is a regular σ-additive complex valued Borel measure on T for $x^* \in X^*$;

(ii) the mapping $x^* \to x^*G(.)$ *of* X^* *into* $C_o(T)^*$ *is weak*- to weak*-continuous;*

(iii) $x^*Uf = \int_T f d(x^*G)$*, for each* $f \in C_o(T)$ *and each* $x^* \in X^*$*; and*

(iv) $||U|| = ||G||(T)$*, where* $||G||$ *is the semi-variation of the vector measure* G*, and by definition*

$$||G||(T) = \sup\{||\sum_{i=1}^{r} \alpha_i G(E_i)|| : |\alpha_i| \leq 1,\ (E_i)_1^r \subset \mathcal{B}(T) \textit{ with } E_i \cap E_j = \emptyset \textit{ for } i \neq j\}$$

Moreover, the vector measure $G : \mathcal{B}(T) \to X^{**}$ *satisfying (i)-(iii) is unique.*

Proof. The proof of the first part of Theorem VI.2.1. of [3] holds here verbatim to prove (i)-(iv) if we replace Ω, $C(\Omega)$ and Σ there by T, $C_o(T)$ and $\mathcal{B}(T)$, respectively.

Now, let us prove the uniqueness of G. If $G_1 : \mathcal{B}(T) \to X^{**}$ is another vector measure satisfying (i)-(iii), then, for each $x^* \in X^*$, the regular σ-additive complex valued Borel measures x^*G and x^*G_1, represent the same bounded inear functional x^*U on $C_o(T)$ and hence $x^*G(E) = x^*G_1(E)$ for $E \in \mathcal{B}(T)$. Since this holds for each $x^* in X^*$, it follows that $G = G_1$.

Now we shall state and prove the principal result.

Theorem 5. *Let* $U : C_o(T) \to X$ *be a bounded linear operator and let us suppose that the Banach space* X *contains no copy of* c_o*. Let* $G_o = G|\mathcal{B}_o(T)$*, where* G *is as in Lemma 4. Then:*

(i) G_o *has range in* X *and* G_o *is* σ*-additive.*

(ii) G *is an* X*-valued* σ*-additive regular Borel measure.*

(iii) $Uf = \int_T f dG, \; f \in C_o(T)$.

(iv) $||U|| = ||G||(T)$.

(v) G is uniquely determined by (ii) and (iii).

Consequently, U is weakly compact.

Proof. By Lemma 4 there exists a unique weak* σ-additive X^{**}-valued Borel measure G on $\mathcal{B}(T)$ such that

$$x^*Uf = \int_T f d(x^*G), \; f \in C_o(T) \tag{1}$$

for each $x^* \in X^*$, x^*G is a regular σ-additive complex valued Borel measure and the mapping $x^* \rightarrow x^*G$ satisfies (ii) of Lemma 4. Moreover, by Lemma 4 (iv), $||U|| = ||G||(T)$.Thus (v) holds.

Let $C \in \mathcal{K}_o(T)$. By Theorem 55.B of Halmos [8] there exists a decreasing sequence (f_n) in $C_o(T)$ such that $f_n \searrow \chi_C$ pointwise on T. Then by (1) and by the Lebesgue dominated convergence theorem

$$x^*G(C) = \lim_n \int_T f_n d(x^*G) = \lim_n x^*Uf_n \tag{2}$$

for each $x^* \in X^*$. Let $Uf_n = x_n$.

For $x^* \in X^*$, x^*G is σ-additive and hence there exist σ-additive positive measures $\mu_{x^*,j} : \mathcal{B}(T) \rightarrow [0, \infty)$, $j = 1, 2, 3, 4$, such that

$$x^*G = (\mu_{x^*,1} - \mu_{x^*,2}) + i(\mu_{x^*,3} - \mu_{x^*,4}).$$

Again by (1) and by the Lebesgue dominated convergence theorem we have

$$\sum_{n=1}^{\infty}|(x^*(x_n - x_{n+1})|= \sum_{n=1}^{\infty}|\int_T (f_n - f_{n+1})d(x^*G)|$$
$$\leq \sum_{j=1}^{4}(\sum_{n=1}^{\infty}\int_T (f_n - f_{n+1})d\mu_{x^*,j})$$
$$\leq \sum_{j=1}^{4}(\int_T f_1 d\mu_{x^*,j} + \mu_{x^*,j}(C))$$
$$< \infty.$$

Hence

$$|x^*(x_1)| + \sum_{n=1}^{\infty}|x^*(x_{n+1} - x_n)| < \infty$$

for each $x^* \in X^*$. Since $c_o \not\subset X$, by Theorem 5 of [2] or by Corollary I.4.5 of [3] the formal series $x_1 + \sum_{n=1}^{\infty}(x_{n+1} - x_n)$ converges unconditionally in norm to some vector $x_o \in X$. In other words, $\lim_n x_n = x_o$ (in norm topology). Then by (2) we have

$$x^*G(C) = \lim_n x^*Uf_n = \lim_n x^*x_n = x^*x_o$$

for each $x^* \in X^*$. Since $G(C) \in X^{**}$, it follows that $G(C) = x_o \in X$. Thus we have proved that $G(\mathcal{K}_o) \subset X$.

Now let $\Sigma = \{E \in \mathcal{B}_o(T) : G(E) \in X\}$. As $\mathcal{K}_o \subset \Sigma$, it follows that $R(\mathcal{K}_o)$, the ring generated by $\mathcal{K}_o$, is contained in Σ. Let $\{E_n\}_1^\infty$ be a monotone sequence in Σ with $\lim_n E_n = E$. When $E_n \nearrow$ put $F_n = E_n \backslash E_{n-1}$ with $E_o = \emptyset$ for $n=$ 1,2,..., and when $E_n \searrow$ put $F_n = E_n \backslash E_{n+1}$ for n=1,2,.... Clearly, $G(F_n) \in X$ for all n. Then $E = \bigcup_1^\infty F_n$ when $E_n \nearrow$ and $E_1 \backslash E = \bigcup_1^\infty F_n$ when $E_n \searrow$. Since x^*G is σ-additive on $\mathcal{B}(T)$, it follows that

$$x^*G(E) = \sum_1^\infty x^*G(F_n) \text{ when } E_n \nearrow$$

and

$$x^*G(E) = x^*G(E_1) - \sum_1^\infty x^*G(F_n) \text{ when } E_n \searrow .$$

Thus in both the cases, $\sum_{n=1}^{\infty} |x^*G(F_n)| < \infty$ for each $x^* \in X^*$. As $c_o \not\subset X$, by Theorem 5 of [2] or by Corollary I.4.5 of [3] the series $\sum_1^{\infty} G(F_n)$ is unconditionally convergent in norm to some vector in X. Then it follows in both the cases that $\lim_n G(E_n) = \omega_o \in X$. Since x^*G is σ-additive,

$$x^*G(E) = \lim_n x^*G(E_n) = x^*\omega_o$$

for all $x^* \in X^*$ and hence $G(E) = \omega_o \in X$. This shows that $E \in \Sigma$ and that Σ is a monotone class. Then by Theorem 6.B of Halmos [8] we conclude that $\Sigma = \mathcal{B}_o(T)$, so that $G_o(\mathcal{B}_o(T)) = G(\mathcal{B}_o(T)) \subset X$.

Since x^*G_o is σ-additive on $\mathcal{B}_o(T)$ for each $x^* \in X^*$ and since the range of G_o is contained in X, by the Orlicz-Pettis theorem it follows that G_o is σ-additive (in the norm topology of X) on $\mathcal{B}_o(T)$. This proves (i).

By Lemma 3, there exists a unique X-valued σ-additive regular Borel measure G' on $\mathcal{B}(T)$ such that $G'|\mathcal{B}_o(T) = G_o$. Since each $f \in C_o(T)$ is G_o-integrable by Theorem 8 of [4], it follows that

$$\int_T f dG_o \in X,\ f \in C_o(T). \tag{3}$$

Then by (1), (3) and the discussion on p. 526 of [4] we have

$$x^* \int_T f dG_o = \int_T f dx^*G_o = \int_T f d(x^*G) = x^*Uf = \int_T f d(x^*G')$$

for $x^* \in X^*$ and $f \in C_o(T)$.

Thus the bounded linear functional X^*U is represented by σ-additive complex valued regular Borel measures x^*G and x^*G' and hence $x^*G = x^*G'$. Since this holds for all $x^* \in X^*$, G' is X-valued and G is X^{**}-valued, we conclude that $G' = G$. This proves (ii).

Since $G_o = G|\mathcal{B}_o(T)$, (iii) follows from (3).

If $\tilde{G} : \mathcal{B}(T) \to X$ satisfies (ii) and (iii), then $x^*\tilde{G}$ and x^*G are σ-additive regular complex valued Borel measures representing the bounded linear functional x^*U and hence $x^*\tilde{G} = x^*G$ for each $x^* \in X^*$. Consequently, by the Hahn-Banach theorem $\tilde{G} = G$. This proves (v).

The weak compactness of U is immediate from (ii) and (iii) and from Theorem VI.1.1 of [3].

This completes the proof.

The following sufficiency part of Theorem 5.3 of Thomas [12], restricted to Banach spaces, follows as a corollary of the above theorem.

Corollary 6. *Every bounded linear operator $U : C_o(T) \to X$ is weakly compact, whenever the Banach space X contains no copy of c_o.*

The following extends Corollary VI.2.16 of [3] to $C_o(T)$.

Corollary 7. *A complemented infinite dimensional subspace of $C_o(T)$ contains a copy of c_o.*

Proof. Suppose X is a complemented infinite dimensional subspace of $C_o(T)$ and let P be a bounded projection of $C_o(T)$ onto X. If $c_o \not\subset X$, then by Corollary 6, P is weakly compact. Then the proof of Corollary VI.2.16 of [3] applies here to show that

P is compact and hence that X is finite dimensional.

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