KRONECKER FUNCTION RING AND STRONG N-RINGS

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Throughout this paper R will be a conmutative ring with identity having total quotient ring T(R). A nonzerodivisor of R is called a **regular element** and an ideal of R is **regular** if it contains a regular element. An ideal I is **faithful** if $0:I = 0$ (for ideals A,B of R, $A:B = \{\mathbf{x} \in R \mid x B \subseteq A\}$) and an ideal is semiregular if it contains a finitely generated faithful ideal. A ring is called a **Prüfer ring** if every finite generated regular ideal is invertible.

Many of the characterizations of Prüfer domains when properly stated for rings with zero divisors yield characterizations of Prüfer rings. For example, see Anderson and Pascual [4], Griffin [6] and Larsen and McCarthy [11, chap. 10]. R is said to be **strongly Prüfer ring** if every finitely generated faithful ideal is locally principal. Strong Prüfer rings were introduced in Anderson *et al.* [3]. In general, our terminology and notation will follows that of Gilmer [7] and Larsen and McCarthy [11].

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Let D be an integrally closed domain with identity and $\{V_\alpha\}$ the set of valuation overrings of D. For an ideal A of D, $\tilde{A} = \bigcap_\alpha AV_\alpha$ is an ideal of D called the **completion** of A. The Kronecker function ring D^K of D is defined to be $D^K = \{f/g \mid f, g \in D[X], \tilde{A}_f \subseteq \tilde{A}_g$ and $D(X) = \{f/g \mid f, g \in D[X], A_g = D\}$. Kronecker function rings have been studied by Prüfer, Krull and Nagata [13]. The generalization for rings with zero divisors has been done by Hinkle and Huckaba [9] and also by Matsuda [12] when the ring R has Property (A) and is a Marot ring. Here, we define the ring R^K for strongly Prüfer rings and for those rings we show that $R(X) = R^K$. Also we define and characterize strong N-rings, a generalization of N-rings.

For $f \in R[X]$, we let A_f be the content of f, that is, if $f = a_0 + a_1X + \ldots + a_nX^n$, then $A_f = (a_0, \ldots, a_n)$, so A_f is the ideal of R generated by the coefficients of f. It is clear that for f, g in $R[X]$, $A_{f+g} \subseteq A_f + A_g$ and that $A_{fg} \subseteq A_fA_g$, also for any element λ in R, we have $A_{\lambda f} = \lambda A_f$. Anderson [2] pointed out that if A_f is locally principal then $A_{fg} = A_fA_g$. Among integral domains with identity, Prüfer domains are characterized by the property that $A_{fg} = A_fA_g$ for each pair f,g of polynomials of $K[X]$. A similar theorem holds for Prüfer rings.

Now, we wish to recall a theorem from Gilmer([7], Corollary 28.3).

PROPOSITION 1. If $f,g \in R[X]$, then there exists a positive integer k such that $A_f^{k+1}A_g = A_f^kA_{fg}$.

If A_f is a cancellation ideal of R, thus for any $g \in T(R)[X]$ we have $A_fA_g = A_{fg}$. Thus for Prüfer rings we have the following.

COROLLARY 2. Let R be a Prüfer ring with total quotient ring T(R). Then for any $f \in T(R)[X]$ such that A_f is regular, we have $A_f A_g = A_{fg}$.

We have a partial converse of Corollary 2.

THEOREM 3. Let R be a ring with identity with total quotient ring T(R). If for all $f,g \in T(R)[X]$, with f or g regular, we have $A_f A_g = A_{fg}$ then R is a Prüfer ring.

Proof. We will show R is integrally closed and for every regular ideal generated by two elements we have $(a,b)^2 = (a^2,b^2)$, then by Theorem 1 of J. Pascual [4] follows that R is a Prüfer ring.

Let b be an element of T(R) integral over R. Then there exists a monic polynomial f(X) in R[X] such that b is a root of f(X). In T(R)[X], $f(X) = (X-b)g(X)$, where $g(X) \in T(R)[X]$. Then we have $R = A_f = A_{(1,b)}A_g$. But g(X) is monic, so $1 \in A_g$. Consequently $b \in (1,b)A_g = R$ so R is integrally closed.

Now if $a,b \in R$ and a is a regular element of R, the equality $(aX+b)(aX-b) = a^2X^2 - b^2$ implies that $A_{(a,b)}A_{(a,b)} = A_{(a^2,b^2)}$ so $(a,b)(a,b) = (a,b)^2 = (a^2,b^2)$. Theorem 1 of Anderson and Pascual [4] implies that R is a Prüfer ring.

Now we shall define *-operations for rings with zero divisors. Let A be a submodule of T(R) with $bA \subseteq R$ for some regular element $b \in R$, then A is called a **fractional ideal** of R. We denote by F(R) the set of all fractional ideals of R.

DEFINITION 4. Let * be a mapping from F(R) into F(R), $* : F(R) \to F(R)$, such that:

(1) For each $A \in F(R)$, we have $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.

(2) For each regular element $b \in T(R)$ and $A \in F(R)$ we have $(b)^* = (b)$ and $(bA)^* = bA^*$.

(3) For each $A \in F(A)$, $(A^*)^* = A^*$.

then * is called a ***-operation** in F(R).

Since each fractional ideal A of R, $A = B/d = Bd^{-1}$ for some ideal B of R, then an *-operation on R is completely determined by its action on the ideals of R.

DEFINITION 5. i) Let * be a *-operation on R, then * is **a.b. *-operation** if

$$(AB)^* \subseteq (AC)^* \text{ implies } B^* \subseteq C^*$$

for A, B and C in $F(R)$ and A finitely generated and regular.

ii) If for B,C finitely generated and A regular finitely generated fractional ideals, we have

$$(AB)^* \subseteq (AC)^* \text{ implies } B^* \subseteq C^*$$

then * is e.a.b. *-operation.

PROPOSITION 6. Let $F \to F^*$ be an *-operation on a ring R. Then for all A,B $F(R)$, we have:

(1) $(\sum_\alpha A_\alpha)^* = (\sum_\alpha A_\alpha^*)^*$ if $\sum A_\alpha$ is a fractional ideal of R.

(2) $(\bigcap_\alpha A_\alpha)^* = (\bigcap_\alpha A_\alpha^*)^*$ if $\bigcap_\alpha A_\alpha^* \neq 0$.

(3) $(AB)^* = (AB^*)^* = (A^*B^*)^*$.

Proof. The same proof as for integral domains (Gilmer [7], Theorem 32.2).

THEOREM 7. Let R be a Prüfer ring and $\{R_\alpha\}$ be the set of all valuation overrings of R. Define the map $* : F(R) \to F(R)$ by $(F)^* = \bigcap_\alpha FR_\alpha$. Then $*$ is a $*$-operation.

Proof. First we note that $F^* = \bigcap_\alpha FR_\alpha$ is a nonempty submodule of $T(R)$ and if d is a regular element of R such that $dF \subseteq R$, then

$$d(\bigcap_\alpha FR_\alpha) = \bigcap_\alpha dFR_\alpha \subseteq \bigcap_\alpha RR_\alpha = \bigcap_\alpha R_\alpha = R.$$

Therefore, $F^* \in F(R)$.

We show that $F \to F^*$ is a $*$-operation.

(1) If a is a regular element of R, then

$(a)^* = \bigcap_\alpha aR_\alpha = (a)(\bigcap_\alpha R_\alpha) = (a)R = (a)$, and if $A \in F(R)$,

$aA^* = a(\bigcap_\alpha AR_\alpha) = \bigcap_\alpha (aA)R_\alpha = (aA)^*$.

(2) It is clear that $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$.

(3) Let $F \in F(R)$, so $F = Ad^{-1}$, where d is a regular element so that $F^* = (Ad^{-1})^* = d^{-1}A^*$ and $F^{**} = A^{**}d^{-1}$. Therefore to show (3) we need to show $A^{**} = A^*$ for A an ideal of R. Since $(A^*)^* = \bigcap_\alpha A^*R_\alpha$, then $A^* \subseteq (A^*)^*$. For each α, AR_α is the extension of an ideal of R, so AR_α is the extension of its contraction to R, $AR_\alpha = (AR_\alpha \cap R)R_\alpha \supseteq A^*R_\alpha$. Then we have $A^* = \bigcap_\alpha AR_\alpha \supseteq \bigcap_\alpha A^*R_\alpha = (A^*)^*$.

Therefore (1), (2) and (3) prove that the conditions of Definition 4 are satisfied so $F^* = \bigcap_\alpha FR_\alpha$ is a $*$-operation.

Moreover, $FR_\alpha = F^*R_\alpha$ for each fractional ideal F.

Furthermore, $F \to F^*$ is a.b. *-operation. Let A,B,C be elements of F(R) with A finitely generated and regular, such that $(AB)^* \subseteq (AC)^*$. Then for each α,

$$ABR_\alpha = (AB)^*R_\alpha \subseteq (AC)^*R_\alpha = ACR_\alpha.$$

Since R is a Prüfer ring and A is finitely generated and regular, then AR_α is a finitely generated regular ideal of R_α, then AR_α is invertible, so that AR_α is a cancellation ideal. Therefore, $BR_\alpha \subseteq CR_\alpha$ for each α, and consequently $B^* \subseteq C^*$.

LEMMA 8. Let R be a strongly Prüfer ring, then for $f,g \in R[X] - \{0\}$ with f regular

$$(A_{fg})^* = (A_fA_g)^*.$$

Proof. Since f is regular, then $0 : A_f = 0$, and then A_f is a semiregular ideal of a strongly Prüfer ring and hence is locally principal, so $A_{fg} = A_fA_g$ and thus $(A_{fg})^* = (A_fA_g)^*$.

LEMMA 9. Let R be a strongly Prüfer ring and let $f,g,h \in R[X]$ such that f is a regular element of $R[X]$, if $(A_fA_g)^* \subseteq (A_fA_h)^*$ then $(A_g)^* \subseteq (A_h)^*$.

Proof. Since R is a strongly Prüfer ring and f is a regular element of $R[X]$, we have $(0:A_f) = 0$, that is, A_f is a semiregular ideal of R. Let $\{R_\alpha\}$ be the set of all valuation overrings of R. Then for each α, $A_fA_gR_\alpha = (A_fA_g)^*R_\alpha \subseteq (A_fA_h)^*R_\alpha = A_fA_hR_\alpha$.

Since R_α is a Prüfer valuation ring, then A_fR_α is a cancellation ideal, then $A_gR_\alpha \subseteq A_hR_\alpha$ for each α. Therefore $(A_g)^* \subseteq (A_h)^*$.

THEOREM 10. Let R be a strongly Prüfer ring and let $R^* = \{0\} \cup \{\frac{f}{g} \mid f,g \in R[X] - \{0\}$ with g regular and $A_f^* \subseteq A_g^*\}$. Then

(1) R^* is a Prüfer ring.

(2) Every finitely generated regular ideal of R^* is principal.

Proof. (1) We have to show first that R^* is well defined. Hence we assume that $f,g,s,t \in R[X] - \{0\}$, and $\frac{f}{g} = \frac{s}{t}$, g,t regular and $A_f^* \subseteq A_g^*$. We show that $A_s^* \subseteq A_t^*$. We have $ft = gs$, so

$$(A_gA_s)^* = (A_{gs})^* = (A_{ft})^* = (A_fA_t)^*$$

$$= (A_f^*A_t)^* \subseteq (A_g^*A_t)^* = (A_gA_t)^*.$$

So $(A_gA_s)^* \subseteq (A_gA_t)^*$.

But A_g is a semi-regular ideal and locally principal, then A_g is a cancellation ideal (Anderson [1], Theorem 3). By Lemma 9, $(A_s)^* \subseteq (A_t)^*$.

If $\frac{f}{g}$ and $\frac{s}{t}$ are nonzero elements of R^*, then $A_{ft-gs} \subseteq A_{ft} + A_{gs}$, so that

$$A_{ft-gs}^* \subseteq (A_{ft} + A_{gs})^* \subseteq (A_{ft}^* + A_{gs}^*)^*$$

$$= ((A_fA_t)^* + (A_gA_s)^*)^*$$

$$= ((A_f^*A_t^*)^* + (A_g^*A_s^*)^*)^*$$

$$\subseteq (A_g^*A_t^*)^* = A_{gt}^*$$

then $\frac{ft-gs}{gt} = \frac{f}{g} - \frac{s}{t}$ is in R^*.

R^* is also closed under multiplication, the proof follows in a similar way.

It is clear that $R \subset R^*$ and $\{X, X^{-1}\} \subseteq R^*$. In particular $R^* \subseteq$ total quotient ring of $R[X]$.

(2) Let A be a finitely generated regular ideal of R^*. We need to show A is principal. We can assume that A is generated by α and β with α regular, say $A = (\alpha, \beta)$, $\alpha = \frac{f}{h}$, $\beta = \frac{g}{h}$, where $f, g, h \in R[X] - \{0\}$. We choose a positive integer n greater than the degree of f. Set $\gamma = \alpha + X^n\beta$, and we prove that $(\alpha, \beta) = (\gamma)$.

The containment $(\gamma) \subseteq (\alpha, \beta)$ is clear.

To show the reverse containment, we need to prove $\frac{\alpha}{\gamma} = \frac{f}{f+X^n g}$, and $\frac{\beta}{\gamma} = \frac{g}{f+X^n g}$ are elements of R^*. Note that $f + X^n g$ is regular. By the choice of n, we have

$$A_{f+X^n g} = A_f + A_g.$$

Then $(A_g)^* \subseteq (A_{f+X^n g})^*$ and $(A_f)^* \subseteq (A_{f+X^n g})^*$.

Therefore, $(\alpha, \beta) = (\gamma)$.

Note that a nonzero element $\frac{a}{b}$ of $T(R)$, where $a, b \in R$, is in R^* if and only of $(a) \subseteq (a)^* \subseteq (b)^* = (b)$, that is, if and only if $\frac{a}{b} \in R$. Therefore $R^* \cap T(R) = R$. Also, if A is a finitely generated regular ideal, say $A = (a_0, a_1, \ldots, a_n)$ then the fact that $AR^* = (a_0 + a_1 X + \ldots + a_n X^n)R^*$ follows in the same way as the proof of (2). Therefore, if $d \in T(R)$, $d \in AR^*$ if and only if $d/(a_0 + a_1 X + \ldots a_n X^n) \in R$ or equivalently, if and only if $(d) \subseteq (d)^* \subseteq (a_0, \ldots, a_n)^*$. Therefore $AR^* \cap T(R) = A^*$.

We state this result in the following.

CORROLARY 5.11. Let R be a strongly Prüfer ring then $R^* \cap T(R) = R$ and if A is a finitely generated regular ideal of R, $AR^* \cap T(R) = A^*$.

The ring R^* defined in Theorem 10 is called the **Kronecker function ring of R with respect to {X}** and the operation *. All the results are valid for an arbitrary set of indeterminates. Therefore, we may extend the Kronecker function ring of R for any set of indeterminates. We denote the Kronecker funtion ring of R with respect to the b-operation, that is, completion by D^K.

DEFINITION 12. Let R be an integrally closed ring and let $\{V_\alpha\}$ the collection of all valuation overrings of R. If I is an ideal of R, then $I' = \bigcap_\alpha IV_\alpha$ is the **completion** of I. If I = I', we say that I is a complete ideal.

PROPOSITION 13. If R is a Prüfer ring, then every ideal of R is complete.

Proof. Let $\{V_\alpha\}$ be the set of all valuation overrings of R. Since R is a Prüfer ring $V_\alpha = R_{[P_\alpha]}$ for $P_\alpha = M_\alpha \cap R$, where M_α is the maximal regular ideal of V_α. Given an ideal I of R, by Gilmer and Huckaba ([8], Lemma 5), we get

$$I' = \bigcap_\alpha IV_\alpha \subseteq IR_{[M_\beta]} = I.$$

Therefore, every ideal I of R is complete.

In [9], Hinkle and Huckaba showed that if R is a Marot ring, then R is a Prüfer ring if and only if every ideal is complete.

We wish now to introduce the ring $R(X)$ and study the relation between $R(X)$ and R. Let $S = \{f \in R[X] \mid A_f = R\}$ then S is a multiplicative closed subset of $R[X]$ and $S = R[X] - (\bigcup_\alpha M_\alpha[X])$, where $\{M_\alpha\}$ is the set of all maximal ideals of R and $M_\alpha[X]$ denotes the ideal of $R[X]$ generated by M_α. The ring $R[X]_S$ is denoted by $R(X)$. For some properties of $R(X)$, the reader is referred to Gilmer ([7], Section 33). The relation between the ideal theory of $R(X)$ and R has been studied by several authors. For integral domains, Arnold [5] showed that the following conditions are equivalent:

(1) R is a Prüfer ring

(2) $R(X)$ is a Prüfer ring

(3) $R(X) = R^K$, where R^K denotes the Kronecker function ring with respect to b-operation.

(4) R^K is a quotient ring of $R[X]$.

Arnold's results have been partially extended to rings with zero divisors by Hinkle and Huckaba [9]. Anderson [1] proved that R is arithmetical of and only if $R(X)$ is arithmetical. Another result of Anderson, Anderson and Markanda [3] is stated here for completeness.

THEOREM 14.

(1) If R is a ring with identity, then $R(X)$ is a Prüfer ring if and only if R is strongly Prüfer.

(2) The following conditions are equivalent.

(a) R is strongly Prüfer.

(b) If $\theta : L(R) \to L(R(X))$ is the map $\theta(I) = IR(X)$, then the regular ideals of $R(X)$ are a subset of the image of θ

(c) θ is a multiplicative lattice isomorphism between the sublattice of semiregular ideals of R and the sublattices of regular ideals of $R(X)$.

The following theorem shows the equality between $R(X)$ and R^K for R strongly Prüfer.

THEOREM 15. If R is a strongly Prüfer ring, then $R(X) = R^K$.

Proof. By definition and Theorem 10, $R^K = \{\frac{f}{g} | f,g \in R[X] - \{0\}$ with g regular and $A_f^* \subseteq A_g^*\}$. Since R is a strongly Prüfer ring, we have that each ideal is complete, so $A_f^* = A_f$ and $A_g^* = A_g$. Then $R^K = \{\frac{f}{g} | f,g \in R[X] - \{0\}$, with $A_f \subseteq A_g$, and $(0:A_g) = (0)\}$.

Since the containment $R(X) \subseteq R^K$ is always true, we have to show only the reverse containment. Let $\frac{f}{g}$ be an element of R^K with $A_f \subseteq A_g$. Since R is a strongly Prüfer ring, A_g is locally principal. Then we have $gR(X) = A_gR(X)$ and $A_gR(X) \supseteq A_fR(X) \supseteq fR(X)$, so $gR(X) \supseteq fR(X)$. Thus $f = gh$ where $h \in R(X)$. Therefore, $\frac{f}{g} = h \in R(X)$ and $R(X) = R^K$.

Larsen [10] defines an **N-ring** R to be a ring R such that for all regular maximal ideals P of R, $R_{[P]}$ is a discrete rank one valuation ring. Griffin [6] proved the following theorem.

THEOREM 16. For a ring R the following are equivalent:

(1) R is an N-ring.

(2) If A,B,C are ideals of R, with A regular the $AB = AC$ implies $B = C$.

(3) R is a Prüfer ring with each regular prime ideal maximal and not idempotent.

(4) The semigroup of regular ideals of R may be embedded in a direct product of ordered cyclic groups.

(5) R is a Prüfer ring such that for any ideal A of R, $\bigcap_{n=1}^{\infty} A^n$ consists entirely of zero divisors.

To extend those results, we introduce a new definition.

DEFINITION 17. A ring R is a **strong N-ring** if the cancellation law holds in R for all semiregular ideals of R.

Then, it follows from the definition that every strong N-ring is an N-ring, and as the following lemma shows, any strong N-ring is strongly Prüfer.

LEMMA 18. If R is a strong N-ring, then R is a strongly Prüfer ring.

Proof. Since the cancellation law holds in R for semiregular ideals, in particular it holds for finitely generated regular ideals of R, so that R is a Prüfer ring. Let A be a finitely generated semiregular ideal of R, then A is a cancellation ideal and for any maximal ideal M, A_M is a cancellation ideal (Gilmer [7], pag. 66 Exercises 5 and 6). Furthermore, A is locally principal, so R is a strongly Prüfer ring.

THEOREM 19. The following conditions are equivalents:

(1) R(X) is an N-ring.

(2) R is a strong N-ring.

(3) R is strongly Prüfer and everly semiregular prime ideal P is maximal and satisfies $P^2 \neq P$.

(4) The semigroup of semiregular ideals of R may be embedded in a direct product of ordered cyclic groups.

Proof. (1) => (2). Let I be a semiregular ideal of R. Then $I(X) = IR(X)$ is a regular ideal of $R(X)$. Now if $IJ = IK$, then we have $IJ(X) = IK(X)$ and $I(X)J(X) = I(X)K(X)$ and since $R(X)$ is an N-ring $I(X)$ is a cancellation ideal, so that $J(X) = K(X)$, and hence $J = J(X) \cap R = K(X) \cap R = K$.

(2) => (3). Assume R is a strong N-ring, then by Lemma 18, R is strongly Prüfer. If P is a semiregular prime ideal of R, then $P \neq P^2$ by the cancellation law. We want to show that any semiregular prime ideal is maximal. Let P be a semiregular prime ideal and $P \subset M$, for M a maximal ideal of R; if we show that $PM = P$ we will get a contradiction. We show this relation locally. Let N be a maximal ideal of R with $N \neq M$, then $MR_N = R_N$ and $(PM)R_N = PR_N MR_N = PR_N$ so $(PM)_N = P_N$.

Assume now that $N = M$. Since P is semiregular then there exists $A \subseteq P$, finitely generated and $(0:A) = (0)$. Let $x \in P$ and $y \in M - P$. Then (A,x,y) is a finitely generated semiregular ideal, so $(A,x,y)_M$ is principal, say, $(A,x,y)_M = (z)_M$ where necessarily $z \in M - P$.

Then $x/1 = \lambda z/1$, where $\lambda \in R_M$. Now $z/1 = x/1 \in R_M$ and $z/1 \notin P_M$, then $\lambda \in R_M$ so that $x/1 \in P_M.M_M$ and then $P_M \subseteq (PM)_M$, and since the other inclusion is always true, we have $P_M = (PM)_M$. Therefore $P = MP$, but P is a cancellation ideal, so $R = M$, a contradiction. Hence P is maximal.

(3) => (1). By Theorem 14, if R is strongly Prüfer, R(X) is a Prüfer ring. Let P(X) be a regular prime ideal R(X). Thus there exists a polynomial $f(X) = a_0 + a_1X + \dots + a_nX^n$ such that $f(X) \in P(X)$ and f(X) is regular, that is, $(A_f:0) = (0)$. Then A_f is a finitely generated semiregular ideal, so A_f is locally principal, then $A_fR(X) = fR(X) \subseteq P(X)$, so $A_f \subseteq P(X) \cap R = P_0$. Hence P_0 is a semiregular prime ideal of R. Thus P_0 is maximal and $P_0^2 \neq P_0$. Then $P_0(X) \subseteq P(X)$ and $P_0(X)$ is maximal and $P(X) = P_0(X) \neq P_0^2(X)$. Therefore, by Theorem 16, R(X) is an N-ring.

(4) => (1). It is clear.

(2) => (4). Since we have already proved the equivalence of (2) and (1), the semigroup of ideals of (RX) may be embedded in a direct product of cyclic groups. By Theorem 14,(c), the map $\theta : L(R) \to L(R(X))$, $\theta(I) = IR(X)$ is an isomorphism between multiplicative lattices, in particular, is a semigroup isomorphism, then the semigroup of semiregular ideals of R may be embedded in a direct product of cyclic groups.

LEMMA 20. If R' is an overring of R, then R'(X) is an overring of R(X).

Proof. Let R' be an overring of R. Then $R'(X) = \{\frac{f}{g} | f,g \in R'[X]$ and $A_g = R'\}$ and the total quotient ring of R(X) is $T(R(X)) = \{\frac{s}{t} | s,t \in R(X)$ and t is regular$\}$. Then if $\frac{w}{t} \in T(R(X))$, $w = \frac{f}{s}$, where $f,s \in R[X]$ with $A_s = R$ and $t = \frac{r}{g}$, where $r,g \in R[X]$ with $A_g = R$ and r regular element of $R[X]$. Then to show that R'(X) is an overring of R(X) it will be enough to prove $R(X) \subseteq R'(X) \subseteq T(R(X))$. If $\frac{f}{g}$ is an element of R'(X), then $f = a_0 + a_1X + \dots + a_nX^n$, $g = b_0 + b_1X +$

$\ldots + b_m X^m$ where $a_i \in R'$, $b_j \in R'$ and $A_g = R'$. Then $f = \frac{1}{s}(a_0'+a_1'X+\ldots+a_n'X^n)$ and $g = \frac{1}{t}(b_0'+b_1'X+\ldots+b_m'X^m)$ where s, a_i', and $b_j' \in R$, $(i=1,\ldots,n,\ j=1,\ldots m)$. Thus

$$\frac{f}{g} = \frac{\dfrac{(a_0'+a_1'X+\ldots+a_n'X^n)st}{s}}{\dfrac{(b_0'+b_1'X+\ldots+b_m'X^m)st}{t}} = \frac{\dfrac{(a_0'+a_1'X+\ldots+a_n'X^n)t}{1}}{\dfrac{(b_0'+b'X+\ldots+b_m'X^m)s}{1}}$$

then $\frac{f}{g} \in T(R(X))$.

Also, if $\frac{f}{g} \in R(X)$, the $A_g = R$, then the ideal generated by the coefficients of g as an ideal of R' is R' since $1 \in A_g$. Then $\frac{f}{g} \in R'(X)$ and consequently $R(X) \subseteq R'(X)$.

Note that we have the following containments:

$$R[X] \subseteq R(X) \subseteq R'(X) \subseteq T(R)(X) \subseteq T(R[X]).$$

PROPOSITION 21. Let R be a strongly Prüfer ring, then any overring of R is also strongly Prüfer.

Proof. Let R' be an overring of R. Then $R(X)$ is a Prüfer ring and by Lemma 20. $R'(X)$ is an overring of $R(X)$, so $R'(X)$ is a Prüfer ring. By Theorem 14, R' is a strongly Prüfer ring.

PROPOSITION 22. Let R be a strong N-ring. Then any overring of R is a strong N-ring.

Proof. Let R' be an overring of R. Then $R'(X)$ is an overring of $R(X)$. Now $R(X)$ is an N-ring and by Larsen ([10], Theorem 4), an overring of an N-ring is an N-ring, so we have $R'(X)$ is an N-ring. Then by Theorem 19, R' is a strong N-ring.

THEOREM 23. Let R be a strong N-ring. Then for any semiregular ideal A of R, R/A is a strong N-ring.

Proof. If R is a strong N-ring, and $\mathcal{N} = N/A$ is a semiregular ideal of R/A, then N is a semiregular ideal of R. Assume we have $\mathcal{N}\mathcal{C} = \mathcal{N}\mathcal{D}$. Let N,D be ideals of R such that $\overline{N} = \mathcal{N}$, $\overline{C} = \mathcal{C}$ and $\overline{D} = \mathcal{D}$, then NC = ND. Since N is semiregular, we have C = D, therefore $\overline{C} = \overline{N}$, that is $\mathcal{C} = \mathcal{D}$. Thus R/A is a strong N-ring.

PROPOSITION 24. Let R be a ring such that R_M is a strong N-ring for every maximal ideal M of R. Then R is a strong N-ring.

Proof. Let A be a semiregular ideal of R. Then there exists a finitely generated ideal B such that $B \subseteq A$ and $(0:B) = 0$. Then $(0:B)_M = (0:B_M) = 0$, so in R_M, A_M is semiregular and by hypothesis R_M is a strong N-ring, so that A_M is a cancellation ideal. By Gilmer ([7], Exercise 6, p. 67), A is a cancellation ideal of R. Hence the cancellation law holds in R for semiregular ideals. Therefore R is a strong N-ring.

THEOREM 25. If R is a strong N-ring, then each semiregular ideal of R with prime radical is a prime power.

Proof. If R is a strong N-ring, then R(X) is a N-ring. Thus any regular ideal of R(X) with a prime radical is a prime power by Larsen ([10], Theorem 1). Let A be a semiregular ideal of R with a prime radical, say $\sqrt{A} = P$. Then P is a semiregular prime ideal of R, and by Theorem 19, P is maximal. Then PR(X) is a maximal ideal of R(X) and AR(X) has PR(X) as its radical, so $AR(X) = P^nR(X)$, and hence $A = AR(X) \cap R = P^nR(X) \cap R = P^n$.

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