A NOTE ABOUT THE MAXIMUM PRINCIPLE

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ABSTRACT. In this paper we give sufficient conditions so that an analytic function having complex domain, range contained in a Banach space and satisfying the maximum principle becomes a constant.

1. PRELIMINARIES. Let Ω be a region of the complex plane and let X be a complex Banach space with dual X^*. A function $f: \Omega \to X$ is called an *analytic function* if and only if $x^* f: \Omega \to \mathbb{C}$ is an analytic function for each $x^* \in X^*$.

In all this work we consider Ω, X and f as above.

For analytic functions with values in Banach spaces the following result is valid (see Hille-Phillips [5] or Dunford-Schwartz [4]).

THEOREM 1.1. *If $f: \Omega \to X$ is an analytic function and there exists $a \in \Omega$ such that $\| f(a) \| \geq \| f(w) \|$ $\forall\, w \in \Omega$, then $\| f \|$ is a constant.*

Theorem 1.1 is known as the maximum principle. We will show in section 3, by means of example, that the maximum principle does not imply that f is constant.

A Banach space X has the *Radon-Nikodym Property* if for each X-valued vector measure μ defined on the Borel sets contained in $[0,1]$, which is of bounded variation and absolutely continuous respect to the Lebesgue's measure m, there is a unique function m - Bochner integrable $g: [0,1] \to X$ such that

$$\mu(E) = \int_E g\,dm \tag{1}$$

for each Borel set $E \in [0,1]$.

An extensive study and a larger bibliography on the Radon-Nikodym Property can be found in Diestel-Uhl [3] . In that reference is proved-among other examples-that ℓ_1, reflexive Banach spaces and dual separable Banach spaces have the Radon-Nikodym Property; however c_0 and ℓ_∞ are Banach spaces which lack this property.

There are many characterizations of Banach spaces with the Radon-Nikodym Property; but in this work we will use the following result obtained by Huff-Morris [6].

THEOREM 1.2. *A real Banach space X has the Radon-Nikodym Property if and only if for each non-empty, bounded and closed A X, the set of all linear and continuous functionals attaining maximum on A is norm-dense in X^*.*

Another geometric property of Banach space used in this work is the following:

THEOREM 1.3. (BISHOP-PHELPS). *If A is a non-empty convex, bounded and closed subset of a real Banach space X, then the set of all linear and continuous functionals attaining maximum on A is norm-dense in X^*.*

The proof of these theorems can be found in Diestel-Uhl [3].

RESULTS. The ideas for the respective proofs of the following theorems 2.1 and 2.2 are contained in the proof of theorem 2.2 of Aurich [1].

THEOREM 2.1. *Let $f: \Omega \to X$ be an analytic function. If there exists $a \in \Omega$ such that $\| f(a) \| \geq \| f(w) \|$ $\forall$ $w \in \Omega$ and $f(\Omega)$ is convex and closed in X, then f is constant.*

PROOF. If $\| f(a) \| = 0$, it is trivial. Suppose that

$\| f(a) \| > 0$ and $x^* \varepsilon X^*$. If x_1^* denotes the real part of x^* and x_1^* attains maximum on $f(\Omega)$, then $x_1^* f$ is constant (see Conway [2] pg. 266) and therefore $x^* f$ is constant.

Suppose that $y^* \varepsilon X^*$ with real part y_1^*. By Bishop-Phelps theorem, given $\varepsilon > 0$ there is a linear and continuous real functional x_1^* which attains its maximum on $f(\Omega)$ and

$$\| x_1^* - y_1^* \| < \frac{\varepsilon}{\|f(a)\|} \tag{2}$$

If we define x^* by means of

$$x^*(x) = x_1^*(x) - i\, x_1^*(ix), \tag{3}$$

Then $x^* f$ is an analytic function whose real part $x_1^* f$ has maximum. Hence $x_1^* f$ is constant. Thus for each $z \varepsilon \Omega$,

$$|y_1^* f(z) - y_1^* f(a)| \leq$$

$$|x_1^* f(a) - y_1^* f(a)| + |x_1^* f(a) - y_1^* f(z)| < \varepsilon. \tag{4}$$

Therefore $y^* f$ is a constant and, by Hahn-Banach theorem, f is constant.

The conclusion of the preceding theorem remains valid if we replace the hypothesis "$f(\Omega)$ is convex" by "X has the Radon-Nikodym Property".

THEOREM 2.2. *Let $f: \Omega \to X$ be an analytic function. If there exists $a \in \Omega$ such that $\| f(a) \| \geq \| f(w) \| \ \forall w \in \Omega$, $f(\Omega)$ is closed in X and X has the Radon-Nikodym Property then f is a constant.*

PROOF. It is similar to the one in theorem 2.1 by changing Bishop-Phelps theorem by Huff-Morris theorem.

3. EXAMPLES.

EXAMPLE 3.1. Let $\Omega = \{z \in \mathbb{C}: |z| < 1\}$ and define $f: \Omega \to C_o$ by

$$f(z) = \{1, z^n, n \geq 1\}. \tag{5}$$

For each $\{\alpha_o, \alpha_1, \ldots, \alpha_n, \ldots\} \in \ell_1 = C_o^*$, we have

$$\{\alpha_o, \alpha_1, \ldots, \alpha_n, \ldots\} f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \tag{6}$$

which is an analytic function. Therefore f is analytic.

It is easy to see that $\| f(z) \|_{C_o} = 1 \quad \forall z \in \Omega$, but f is not constant. This last conclusion is due to the fact that although $f(\Omega)$ is closed it is not convex.

It is well known that C_o lacks the Radon-Nikodym property. A new way to prove this fact is by using example 3.1

together theorem 2.2 from the preceding section.

EXAMPLE 3.2. Let Ω be as example 3.1 and define $f: \Omega \to C_o$ by $f(z) = \{1, \frac{z}{n}, n \geq 1\}$. The function f defined above is non constant but $\| f(z) \|_{C_o} = 1 \quad \forall z \in \Omega$.

On the other hands, it is easy to see that $f(\Omega)$ is convex but not closed and that f is an analytic function. This proves that the hypothesis "$f(\Omega)$ closed" cannot be removed in theorem 2.1 of precedent section.

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