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A NOTE ON REGULAR EXTENSIONS

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ABSTRACT

Let μ_o be a Baire measure on T, a locally compact Hausdorff space. If μ is the regular Borel extension of μ_o as given in Halmos [4] and $\hat{\mu}$ is the regular weakly Borel extension of μ_o obtained via the Riesz representation theorem given in Rudin [6] or Hewitt and Stromberg [5] , then an explicit description of $\hat{\mu}$ in terms of the outer measure μ^* induced by μ is given, thereby clarifying the implicit connection between the results of Halmos [4] and those of Rudin [6] .

RESUMEN

Sea μ_o una medida de Baire en T, un espacio localmente compacto y de Hausdorff. Si μ es la extensión regular Boreleana de μ_o como se dió en Halmos [4] y $\hat{\mu}$ es la extensión regular débilmente Boreleana de μ_o deducida a través del teorema de representación de Riesz dado en Rudin [6] o Hewitt y Stromberg [5] , entonces damos una descripción explicita de $\hat{\mu}$ en términos de la medida exterior μ^* inducida por μ. Este estudio aclara las conexiones implicitas entre los resultados de Halmos [4] y los de Rudin [6] .

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Let T be a locally compact Hausdorff space. $\mathcal{U}$ (respy. $\mathcal{U}_o$) is the family of the open sets (respy. open F_σ sets) in T; $\mathcal{C}$ (respy. $\mathcal{C}_o$) is that of the compact sets (respy. compact G_δ sets) in T. For a family $\mathcal{F}$ of sets in T let $\mathcal{S}(\mathcal{F})$ denote the σ-ring generated by $\mathcal{F}$. Let $\mathcal{B}_o = \mathcal{S}(\mathcal{C}_o)$, $\mathcal{B} = \mathcal{S}(\mathcal{C})$ and $W = \mathcal{S}(\mathcal{U})$. The members of $\mathcal{B}_o$ (resp. $\mathcal{B}$, W) are called Baire (respy. Borel, weakly Borel) sets of T (vide [1]). A measure is a positive measure. A measure m on $\mathcal{B}_o$ (respy. $\mathcal{B}$, W) is called a Baire (respy. Borel, weakly Borel) measure if $m(C) < \infty$ for $C \in \mathcal{C}_o$ (respy. $C \in \mathcal{C}$).

For a ring $\mathcal{R}$ of sets in T, $H(\mathcal{R})$ denotes the hereditary σ-ring generated by $\mathcal{R}$. If μ is a measure on $\mathcal{R}$ let μ^* be the outer measure induced by μ on $H(\mathcal{R})$ and let M_{μ^*} be the σ-ring of all μ^*-measurable sets in $H(\mathcal{R})$. Let $\tau(\mu) = \{ E \subset T: E \cap A \in M_{\mu^*}, A \in M_{\mu^*} \}$. It is known from [3] that $\tau(\mu)$ is a σ-algebra, $\tau(\mu) \supset M_{\mu^*}$ and the set function

$$\tilde{\mu}^*(E) = \sup \{ \mu^*(F) : E \supset F \in M_{\mu^*} \} , \quad E \in \tau(\mu)$$

is a measure on $\tau(\mu)$ and $\tilde{\mu}^* | M_{\mu^*} = \mu^*$. Hence by abuse of notation we shall denote $\tilde{\mu}^*$ by μ^* and the measure μ^* on $\tau(\mu)$ is said to be induced by μ.

Definition 1: Let m be a weakly Borel measure on W. We say that m is regular if

(i) $m(E) = \inf \{ m(U) : E \subset U \in \mathcal{U} \}$, $E \in W$

and

(ii) $m(U) = \sup \{ m(C) : U \supset C \in \mathcal{C} \}$, $U \in \mathcal{U}$.

A Borel (respy. Baire) measure m is said to be regular if

(i') $m(E) = \sup\{m(C): E \supset C \in \hat{\mathcal{C}}\}$,

(ii') $m(E) = \inf\{m(U): E \subset U \in \hat{\mathcal{B}} \cap \mathcal{U}\}$,

for $E \in \hat{\mathcal{B}}$, where $\hat{\mathcal{B}} = \mathcal{B}$ (respy. $\mathcal{B}_o$) and $\hat{\mathcal{C}} = \mathcal{C}$ (respy. $\mathcal{C}_o$).

By Theorem 52.G of [4] every Baire measure μ_o on T is regular and by Theorem 54.D of [4] μ_o can be extended (uniquely) to a regular Borel measure μ on T. Let $\Lambda f = \int_T f\,d\mu_o$, $f \in C_c(T)$. Then Λ is a positive linear form on $C_c(T)$ and hence by the Riesz representation theorem given in Chapter 2 of [6] or in Chapter 3 of [5] there exists a regular weakly Borel measure $\hat{\mu}$ on T such that

$$\Lambda f = \int_T f\,d\hat{\mu}, \quad f \in C_c(T).$$

Then $\hat{\mu} \mid \mathcal{B}$ is a regular Borel extension of μ_o and hence $\hat{\mu} \mid \mathcal{B} = \mu$. Thus $\hat{\mu}$ is a regular weakly Borel extension of μ_o and μ and this extension is unique.

On the other hand if μ^* on $\tau(\mu)$ is the measure induced by the regular Borel extension μ of μ_o, what is the relation between μ^* and $\hat{\mu}$?. In this note we obtain $\hat{\mu}$ in terms of μ^* and observe that $\mu^* \neq \hat{\mu}$ in general.

In the sequel μ_o is a Baire measure on T, μ and $\hat{\mu}$ are its unique regular Borel and weakly Borel extensions respectively.

The following result is immediate from Theorems 1 and 2 of §16 in Dinculeanu [3].

Lemma 2: (Dinculeanu [3]). Let μ^* be the measure induced by μ on $\tau(\mu)$. Then:

(i) $\mu^* \mid \mathcal{B} = \mu$.

(ii) $\mathcal{U} \subset \tau(\mu)$ and hence $W \subset \tau(\mu)$.

(iii) For $A \in \mathcal{B}$, $\mu(A) = \inf\{\mu^*(U): A \subset U \in \mathcal{U} \cap \mathcal{B}\}$.

(iv) For $A \in \tau(\mu)$, $\mu^*(A) = \sup\{\mu(K): A \supset K \in \mathcal{C}\}$.

Definition 3: Let $\lambda: W \to [o, \infty]$ be given by

$$\lambda(E) = \inf\{\mu^*(U): E \subset U \in \mathcal{U}\}.$$

In oder to show that $\lambda = \hat{\mu}$ we give the following lemmas.

Lemma 4:

(i) $\lambda \mid \mathcal{B} = \mu$.

(ii) $\lambda \mid \mathcal{U} = \mu^* \mid \mathcal{U}$ and hence $\lambda(E) = \inf\{\lambda(U): E \subset U \in \mathcal{U}\}$.

(iii) $\lambda(U) = \sup\{\lambda(K): U \supset K \in \mathcal{C}\}$, $U \in \mathcal{U}$.

(iv) λ is countably subadditive.

Proof.

(i) By Lemma 2(ii) we have $\mu(A) = \inf\{\mu^*(U): A \subset U \in \mathcal{U} \cap \mathcal{B}\}$ $\geqq \inf\{\mu^*(U): A \subset U \in \mathcal{U}\} \geqq \mu^*(A) = \mu(A)$ for $A \in \mathcal{B}$. Hence $\lambda \mid \mathcal{B} = \mu$.

(ii) Trivial.

(iii) As $\mathcal{U} \subset \tau(\mu)$, (iii) follows from (i) and Lemma 2(iv).

(iv) Let $E = \bigcup_1^\infty E_n$, $E_n \in W$. It suffices to assume $\lambda(E_n) < \infty$ for all n. Let $\varepsilon > 0$. Then by (ii) for each E_n there exists $U_n \in \mathcal{U}$ with $E_n \subset U_n$ and $\lambda(U_n) < \lambda(E_n) + \frac{\varepsilon}{2^n}$. Let $U = \bigcup_1^\infty U_n$. Then $E \subset U \in \mathcal{U}$ and by (ii) we have

$$\lambda(E) \leqq \lambda(U) = \mu^*(U) \leqq \sum_1^\infty \mu^*(U_n) < \sum_1^\infty \lambda(E_n) + \varepsilon .$$

Hence (iv) holds.

Lemma 5. Let $\Sigma_F = \{ E \in W: \lambda(E) < \infty \text{ and } \lambda(E) = \sup_{\substack{K \subset E \\ K \in \mathcal{C}}} \lambda(K) \}$. Then:

(i) For $E \in W$ and $K \in \mathcal{C}$, $E \cap K \in \Sigma_F$.

(ii) $\Sigma_F = \{ E \in W: \lambda(E) < \infty \}$.

(iii) λ is countably additive on W.

Proof.

(i) Clearly, $W = \{ E \in W: E \cap K \in \mathcal{B},\ K \in \mathcal{C}\}$. Then for $E \in W$ and $K \in \mathcal{C}$, $\lambda(E \cap K) = \mu(E \cap K) < \infty$ by Lemma 4(i). Consequently, $E \cap K \in \Sigma_F$ since μ is a regular Borel measure and $\lambda | \mathcal{B} = \mu$.

(ii) Let $E \in W$ with $\lambda(E) < \infty$. Then there exists $U \in \mathcal{U}$ with $E \subset U$ and $\mu^*(U) < \infty$. Let $\varepsilon > 0$. By Lemma 4(iii) there exists $K \in \mathcal{C}$ with $K \subset U$ and $\lambda(U) < \lambda(K) + \varepsilon/_2$. Consequently, $0 < \lambda(U) - \lambda(K) = \mu^*(U) - \mu^*(K) = \mu^*(U \setminus K) = \lambda(U \setminus K) < \frac{\varepsilon}{2}$ since $\mathcal{U} \subset \tau(\mu)$, $\mu^* | \mathcal{U} = \lambda | \mathcal{U}$ and μ^* is subtractive on $\tau(\mu)$.

On the other hand, by (i) $E \cap K \in \Sigma_F$ and hence there exists $K_1 \in \mathcal{C}$ with $K_1 \subset E \cap K$ and $\lambda(E \cap K) < \lambda(K_1) + \varepsilon/_2$. Therefore,

$$\lambda(E) \leq \lambda(E \cap K) + \lambda(U \setminus K)$$

$$< \lambda(K_1) + \varepsilon$$

since λ is subadditive. Hence $E \in \Sigma_F$.

(iii) First we shall show that λ is additive. Let $E_1, E_2 \in W$ with $E_1 \cap E_2 = \emptyset$. As λ is subadditive we shall assume $\lambda(E_1 \cup E_2) < \infty$. Then, given $\varepsilon > o$, by (ii) there exist $K_i \subset E_i$, $K_i \in \mathcal{C}$ with $\lambda(E_i) < \lambda(K_i) + \frac{\varepsilon}{2}$, $i = 1,2$.

As $\lambda | \mathcal{B} = \mu$, we have

$$\lambda(E_1) + \lambda(E_2) < \mu(K_1 \cup K_2) + \varepsilon \leq \lambda(E_1 \cup E_2) + \varepsilon$$

so that $\lambda(E_1) + \lambda(E_2) \leq \lambda(E_1 \cup E_2)$. Now the result (iii)holds by Lemma 4(iv).

Theorem 6: Let μ_o be a Baire measure on T with its regular Borel extension μ. If μ^* is the measure induced by μ on $\tau(\mu)$, then the set function λ gives by

$$\lambda(E) = \inf \{\mu^*(U) : E \subset U \in \mathcal{U},\}, E \in W$$

is the unique regular weakly Borel extension of μ_o and hence $\hat{\mu} = \lambda$. Besides, $\hat{\mu}$ does not coincide with $\mu^* | W$ in general.

Proof: By Lemma 5(iii) λ is a measure on W. By Lemma 4(i) λ

is an extension of μ and hence $\lambda(C) < \infty$ for $C \in \mathcal{C}$. Por Lemma 4(iii) and by the definition of λ, λ is regular as a weakly Borel measure. If λ' in another regular weakly Borel extensión clearly $\lambda' = \lambda$ and hence λ is unique and coincides with $\hat{\mu}$.

As $\mu^*(E) = \sup\{\mu^*(K) : E \supset K \in \mathcal{C}\}$ for $E \in W$ by Lemma 2(iv) and since $\hat{\mu}(E) = \sup\{\hat{\mu}(K) : E \supset K \in C\}$ does not hold for all $E \in W$ in general (vide Exercise 16, Chapter 2 of [6]), we conclude that $\hat{\mu} \neq \mu^*$ in general.

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