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ORTHOGONAL AND BOUNDED ORTHOGONAL SPECTRAL REPRESENTATIONS

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ABSTRACT

The concepts of orthogonal spectral representations (OTSRs, in abbreviation), bounded orthogonal spectral representations (BOTSRs, in abbreviation) and BOTSRs with countable multiplicities (COBOTSRs, in abbreviation) are introduced and is obtained a complete set of unitary invariants for a spectral measure E(.) in terms of these representations. Also is shown that E(.) has the generalized CGS-property (respy. CGS-property) in a Hilbert space H if and only if H has a BOTSR (respy. COBOTSR) relative to E(.). Besides, is studied the inter-relation between orde - red spectral representations (or ordered spectral decompositions) and COBOTSRs of H relative to E(.) when E(.) has the CGS-proper-

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ty in H and the ordered spectral decompositions are classified in four types and each of them is characterized in terms of the multiplicity set of E(.).

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Dunford and Schwartz obtained in [2] a complete set of unitary invariants of an operator T on a separable Hilbert space H_0 in terms of ordered spectral representations (OSRs, in abbreviation) of H_0 relative to T, when T is self-adjoint or is bounded and normal. In [7] we extend the results of [2] to spectral measures E(.) on an arbitrary Hilbert space H, when E(.) has the CGS-property in H, thereby obtaining a complete set of unitary invariants of E(.) in terms of OSRs (and ordered spectral decompositions) of H relative to E(.). In this work we introduce the concept of orthogonal spectral representations (OTSRS, in abbreviation), study their properties and obtain a complete set of

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unitary invariants of an arbitrary spectral measure E(.) in terms of these representations. Thus we obtain results analogous to Theorems 9.8 and 9.9 of [7] (or to Theorems X.5.10, X.5.12 and XII.3.16 of [2]) for OTSRs. Besides, is given an alternate proof for the main theorem of Halmos on p. 109 of [3].

We study two particular types of OTSRS, called BOTSRs and COBOTSRs, of H relative to E(.) and study their properties. We show that E(.) has the generalized CGS-property (respy. -CGS-property) in H if and only if every OTSR of H relative to E(.) is a BOTSR (respy. a COBOTSR). Various complete sets of unitary invariants of E(.) are given when E(.) has the generalized CGS-property and is given a theorem on the equivalence of two BOTSRs.

We also deduce the extensions of some of the results of Plesner and Rohlin [10] for spectral measures. This study obviously sheds more light on the work of Halmos [3] and on that of [10] when the spectral measure has the CGS-property or the generalized CGS-property.

In [7] we introduced the concepts of OSD-multiplicity, OSR-multilplicity and total multiplicity of a spectral measura E(.) with the CGS-property and showed that all of them are the same. Here we introduce the concept of total H-multiplicity (H stands for Halmos) of a spectral measure E(.) and prove that it coincides with its total multiplicity when it has the CGS-property. Given a COBOTSR of H relative to E(.) we construct in a canonical way an ordered spectral decomposition (OSD, in abbreviation) (equivalently an OSR) of H relative to E(.) and vice versa. Finally, we classify the OSDs (respy. OSRs) of H relative to E(.) in four types and characterize each one of them in terms of the multiplicity sets of E(.).

1. PRELIMINARIES. In this section we fix the terminology and notations and give some definitions and results from the literature which are needed in the sequel.

S is a σ-algebra of subsets of a set X ($\neq \phi$). H, H_1 and H_2 are (complex) Hilbert spaces and E(.), $E_1(.)$ and $E_2(.)$ are spectral measures on S with values in projections of H, H_1 and H_2 respectively. The closed subspace spanned by a subset $\mathfrak{X}$ is denoted by $[\mathfrak{X}]$. For a vector x ε H, $Z(x) = [E(\sigma)x, \sigma \in S]$. Similarly, $Z_i(x_i) = [E_i(\sigma)\, x_i, \sigma \in S]$, $x_i \in H_i$, i=1,2. $\Sigma \oplus$ denotes either the orthogonal sum of a family of closed subspaces or of Hilbert spaces, as the case may be.

W, W_1 and W_2 are respectively the von Neumann algebras generated by the ranges of E(.), $E_1(.)$ and $E_2(.)$. W' (respy. W_i') is the commutant of W (respy. W_i). If $W' = \Sigma \oplus W' Q_n$ is the type I_n direct sum decomposition of W', then the central projections $Q_n(\neq 0)$ are unique (such that $W' Q_n$ is of type I_n) and in the sequel Q_n will denote these central projections. Similarly, $Q_n^{(i)}$, are defined with respect to W_i' for i=1,2. For a projection $P' \varepsilon W'$, $C_{P'}$ denotes the central support of P'. Other terminology in von Neumann algebras is standard and we follow Dixmier [1].

As was observed in [6] a projection P' in W' is abelian if and only if P' is a row projection in the sense of Halmos [3] and the column C(P') generated by P' as in [3] is the same as $C_{P'}$. An operator T on H is a linear transformation with domain and range contained in H and is not necessarily bounded.

NOTATION 1.1. Let P be a projection in W. The multiplicity (respy. uniform multiplicity) of P in the sense of Halmos [3] will be referred to as its H-multiplicity (respy. UH-multiplicity) relative to E(.).

As was noted in [6] Theorem 64.4 of Halmos [3] can be interpreted as follows:

THEOREM 1.2. A non-zero projection F in W has UH-multiplicity n if and only if there exists an orthogonal family $\{E_\alpha'\}_{\alpha \varepsilon J}$ of abelian projections in W' such that card. J = n, $C_{E_\alpha'} = F$ and $\sum_{\alpha \varepsilon J} E_\alpha' = F$. In other words, F has UH-multiplicity n if and only if W'F is of type I_n.

Consequently, the following proposition is immediate.

PROPOSITION 1.3. A non-zero projection P in W has UH-multiplicity n if and only if $P \leq Q_n$.

$\rho(x)$ denotes the measure $\|E(.)x\|^2$ for $x \in H$. Similarly, $\rho_i(x_i) = \|E_i(.)x_i\|^2$, $x_i \in H_i$, $i=1,2$. Σ denotes the set of all finite (positive) measures on S. For $\mu_1, \mu_2 \in \Sigma$ we write $\mu_1 \equiv \mu_2$ if $\mu_1 << \mu_2$ and $\mu_2 << \mu_1$. Then '$\equiv$' is an equivalence relation on Σ. We say μ_1 is orthogonal to μ_2 ($\mu_1 \perp \mu_2$, in symbols) if $\nu << \mu_1$ and $\nu << \mu_2$, $\nu \in \Sigma$ then $\nu = 0$.

For $x \in H$, $[Wx] = [Ax: A \in W]$ and, sometimes, also denotes the orthogonal projection with the range $[Wx]$. For $\mu \in \Sigma$ the projection $C(\mu)$ (respy. $C_i(\mu)$) relative to $E(.)$ (respy. $E_i(.)$) is defined as the orthogonal projection on the subspace $\{x \in H: \rho(x) << \mu\}$ (respy. $\{x: H_i: \rho_i(x) << \mu\}$) and is well-known that $C(\mu) \in W$. The multiplicity $u(\mu)$ (respy. $\mu_i(\mu)$) of μ relative to $E(.)$ (respy. $E_i(.)$) is defined by

$$u(\mu) = \min \{\text{H-multiplicity of } C(\nu): 0 \neq \nu << \mu\}$$

if $\mu \neq 0$ and $u(0) = 0$ (respy. $u_i(\mu) = \min\{\text{H-multiplicity of } C_i(\nu): 0 \neq \nu << \mu\}$ if $\mu \neq 0$ and $u_i(0) = 0$). μ is said to have uniform multiplicity $u(\mu)$ if $u(\mu) = u(\nu)$ for all $\nu \in \Sigma$ with $0 \neq \nu << \mu$.

2. ORTHOGONAL SPECTRAL REPRESENTATIONS (OTSRs). In this section we introduce the definition of an OTSR of H relative to E(.) and obtain several characterizations of the measure family of an OTSR. Finally, we show that for a given spectral measure E(.) on H there always exists an OTSR of H relative to E(.).

LEMMA 2.1. Let $\mu \in \Sigma$. Then:

(i) If $C(\mu) = 0$, then μ has uniform multiplicity zero relative E(.).

(ii) If μ has non-zero uniform multiplicity $u(\mu)$ then $u(\mu)$ is also the H-multiplicity of $C(\mu)$.

(iii) In (ii), $C(\mu)$ has UH-multiplicity $u(\mu)$.

PROOF.

(i) By Theorem 66.3 of [3], $C(\nu) \leq C(\mu) = 0$ if $\nu << \mu$ and hence (i) holds.

(ii) This is immediate from the definition of uniform multiplicity of μ and the fact that H-multiplicity of $C(\nu) \geq$ H-multiplicity of $C(\mu)$ if $0 \neq \nu << \mu$ and $C(\nu) \neq 0$.

(iii) By hypothesis and by (i), $C(\mu) \neq 0$. Let x be a non-zero vector in $C(\mu)$ H. Suppose Q is a non-zero projection in W such that $Q \leq C_{[Wx]}$. By Theorem 66.2 of [3] $C_{[Wx]} = C(\rho(x))$ and hence by (ii), the H-multiplicity of $C_{[Wx]}$ is $u(\mu)$.

Besides, $Q = QC_{[Wx]} = Q\,[W'x] = [W'Qx] = C_{[WQx]} = C(\rho(Qx))$ by Corollary 2 of Proposition I.1.7 of [1] and by Theorem 66.2 of [3]. Therefore, the H-multiplicity of Q coincides with $u(\rho(Qx)) = u(\mu)$ and hence $C_{[Wx]}$ has UH-multiplicity $u(\mu)$. Now, let $\{x_j\}_{j\varepsilon J}$ be a maximal orthogonal family of non-zero vectors in $C(\mu)\,H$ such that $C_{[Wx_j]}\,C_{[Wx_{j'}]} = 0$ for $j \neq j'$. Then $C(\mu) = \sum_{j\varepsilon J} C_{[Wx_j]}$ and consequently, by Theorem 64.3 of [3] we conclude that $C(\mu)$ has UH-multiplicity $u(\mu)$.

DEFINITION 2.2. An isomorphism U from H onto $\tilde{H} = \sum_{j\varepsilon J} \oplus \sum_{u_j} \oplus L_2(\mathbf{X}, S, \mu_j)$ is called an orthogonal spectral representation (an OTSR, in abbreviation) of H relative to E(.) if

(i) $\{\mu_j\}_{j\varepsilon J}$ is an orthogonal family in Σ;

(ii) each μ_j has uniform multiplicity $u(\mu_j) = u_j > 0$ and

(iii) $UE(.)\,U^{-1} = \tilde{\mathbf{E}}(.)$, where

$$\tilde{E}(.)\,(f_{jk})_{\substack{k\,\varepsilon\, I_j \\ j\,\varepsilon\, J}} = (\chi_{(.)}\,f_{jk})_{\substack{k\,\varepsilon\, I_j \\ j\,\varepsilon\, J}}$$

for $(f_{jk})_{\substack{k\varepsilon I_j \\ j\varepsilon J}} \varepsilon\, \tilde{H}$, with card. $I_j = u_j$.

(Here $\sum_{u_j} \oplus L_2(X,S,\mu_j) = \sum_{k\varepsilon I_j} \oplus L_2(X,S,\mu_{jk})$, where card. $I_j = u_j$ and $\mu_{jk} = \mu_j$, $k\,\varepsilon I_j$).

The set $\{\mu_j\}_{j\varepsilon J}$ is called the measure family of the representation U. By M_E we denote the set $\{n: Q_n \neq 0\}$ and we call M_E the multiplicity set of E(.). We say that U is an OTSR of H relative to a normal operator (bounded or not) T on H if it is so relative to the resolution of the identity of T.

PROPOSITION 2.3. Let U be an OTSR of H relative to E(.) with the measure family $\{\mu_j\}_{j\varepsilon J}$. Suppose that each μ_j has uniform multiplicity u_j (relative to E(.)). Then the following assertions hold:

(i) $\{C(\mu_j)\}_{j\varepsilon J}$ is an orthogonal family of non-zero projections in W.

(ii) $\sum_{j\varepsilon J} C(\mu_j) = I$.

(iii) For $\mu \varepsilon \Sigma$ with $C(\mu) \neq 0$,

$$C(\mu) = C(\bigvee_{j\varepsilon J} (\mu \wedge \mu_j)).$$

(iv) For $\mu \varepsilon \Sigma$ let $u(\mu)$ be the multiplicity of μ relative to E(.). If $u(\mu) > 0$, then

$$\mu \equiv \bigvee_{j\varepsilon J} (\mu \wedge \mu_j).$$

(v) $M_E = \{u_j: j \varepsilon J\}$ and coincides with the set of all UH-multiplicities of non-zero projections in W.

(vi) $Q_n = \sum_{u_j=n} C(\mu_j)$, $n \in M_E$.

PROOF.

(i) By Lemma 2.1, the H-multiplicity of $C(\mu_j)$ is the same as u_j (>0) and hence $C(\mu_j) \neq 0$, $j \in J$. By Theorem 66.3 of [3] $C(\mu_j) \perp C(\mu_j{}')$ for $j \neq j'$.

(ii) If possible, let $Q = I - \sum_{j\in J} C(\mu_j) \neq 0$. Then, by Theorem 66.2 of [3], for a non-zero vector $x \in QH$ we have $C(\rho(x)) = C_{[Wx]} \leq Q$. Then by Theorem 66.1 of [3], $C(\rho(x)) \wedge \mu_j) = 0$ for $j \in J$. On the other hand, if $0 \neq \nu << \rho(x) \wedge \mu_j$, then by Theorem 65.3 of [3] there exists $y \in C_{[Wx]}$ such that $\nu = \rho(y)$ so that $C(\nu) \neq 0$.

Therefore, we conclude that $\rho(x) \perp \mu_j$, $j \in J$.

If $Ux = f \in \tilde{H}$, then by hypothesis

$$0 \neq \rho(x) = \| E(.)x\|^2 = \| UE(.)x\|^2 = \| \tilde{E}(.)f\|^2 .$$

If $f = (f_{jk})_{k\in I_j, j\in J}$, card. $I_j = u_j$, then there exists $j_o \in J$ and $k_o \in I_{j_o}$ such that $f_{j_o k_o} \neq 0$ μ_{j_o} - a.e. Let $g = (g_{jk})_{k\in I_j, j\in J} \in \tilde{H}$, where $g_{j_o k_o} = f_{j_o k_o}$ and $g_{jk} = 0$ otherwise. Clearly, $\| \tilde{E}(.)g\|^2 << \| \tilde{E}(.)\ f\|^2 = \rho(x)$. Let $U^{-1} g = y$. Then $y \neq 0$, $\| \tilde{E}(.)g\|^2 = \rho(y)$ and $\rho(y) << \rho(x)$.

On the other hand, $\|\tilde{E}(.)g\|^2 << \mu_{j_0}$ and hence $\rho(x) \wedge \mu_{j_0} \neq 0$. This contradiction proves (ii).

(iii) By (ii) we have $C(\mu) = \sum_{j\varepsilon J} C(\mu)\ C(\mu_j)$. Since $\{\mu \wedge \mu_j\}_{j\varepsilon J}$ is a bounded orthogonal family, the set $J_0 = \{j: \mu \wedge \mu_j \neq 0\}$ is countable. Then by Theorems 66.3 and 66.5 of [3] and by hypothesis we have

$$0 \neq C(\mu) = \sum_{j\varepsilon J_0} C(\mu \wedge \mu_j) = C(\bigvee_{j\varepsilon J_0} (\mu \wedge \mu_j)) = C(\bigvee_{j\varepsilon J} (\mu \wedge \mu_j)).$$

(iv) Let $\nu \equiv \bigvee_{j\varepsilon J} (\mu \wedge \mu_j)$. Clearly, $\nu << \mu$. If $\nu \not\equiv \mu$, then by Theorem 48.2 of [3] there exists $0 \neq \nu_0 << \mu$ such that $\nu \perp \nu_0$ and $\nu \vee \nu_0 \equiv \mu$. Then by Theorem 66.3 of [3] and by (iii) we have

$$0 = C(\nu)\ C(\nu_0) = C(\mu)\ C(\nu_0) = C(\mu \wedge \nu_0) = C(\nu_0).$$

Consequently, $u(\mu) = 0$. This contradiction to the hypothesis that $u(\mu) > 0$ implies that $\nu \equiv \mu$.

(v) Since μ_j has uniform multiplicity $u_j > 0$, by Proposition 1.3 and Lemma 2.1 (iii), clearly $\{u_j: j \varepsilon J\} \subset M_E$. Con - versely, let $n \varepsilon M_E$. By (ii), $Q_n = \sum_{j\varepsilon J} C(\mu_j)\ Q_n \neq 0$ so that there exists $j_0 \varepsilon J$ such that $C(\mu_{j_0})\ Q_n \neq 0$. As $C(\mu_{j_0})$ has UH-multiplicity u_{j_0} by Lemma 2.1 (iii) and Q_n has UH-multiplicity n by Proposition 1.3, we conclude that $u_{j_o} = n$. The last part is evident from Proposition 1.3.

(vi) Let $P_n = \sum_{u_j=n} C(\mu_j)$, $n \in M_E$. Then by Lemma 2.1 (iii) and by Theorem 64.3 of [3], P_n has UH-multiplicity n and hence, by Proposition 1.3, $P_n \leq Q_n$. If $P_n \neq Q_n$, then by (ii) $(Q_n - P_n)\, C(\mu_{j_o}) \neq 0$ for some $j_o \in J \setminus \{j: u_j = n\}$ which is impossible by Proposition 1.3 and Lemma 2.1 (iii) as $n \neq u_{j_o}$.

The following theorem gives several characterizations of the measure family of an OTSR of H.

THEOREM 2.4. Let $\{\mu_j\}_{j\varepsilon J}$ be an orthogonal family of non-zero finite measures on S, each μ_j having uniform multiplicity $u_j > 0$ relative to E(.). Then the following statements are equivalent.

(i) $\{\mu_j\}_{j\varepsilon J}$ is the measure family of an OTSR of H relative to E(.).

(ii) $\{C(\mu_j)\}_{j\varepsilon J}$ is an orthogonal family of non-zero projections and $\sum_{j\varepsilon J} C(\mu_j) = I$.

(iii) For $\mu \in \Sigma$ with $C(\mu) \neq 0$, $C(\mu) = C(\bigvee_{j\varepsilon J} (\mu \wedge \mu_j))$.

(iv) For $\mu \in \Sigma$ with $u(\mu) > 0$, $\mu \equiv \bigvee_{j\varepsilon J} (\mu \wedge \mu_j)$ and if $u(\mu) = 0$ with $C(\mu) \neq 0$, then $C(\mu) = C(\bigvee(\mu \wedge \mu_j))$.

(v) $Q_n = \sum_{u_j=n} C(\mu_j)$, $n \in M_E$.

PROOF.By Proposition 2.3 it is clear that (i) => (ii), (i) =>(iii), (i) =>(iv) and (i) =>(v). Now we prove the reverse implications by proving (ii) =>(i), (iii) =>(ii), (iv) =>(iii) and (v) =>(ii).

(ii) =>(i) Let $n \in M_E$. As μ_j has uniform multiplicity $u_j > 0$, as in the proof of (vi) in Proposition 2.3, we have $Q_n = \sum_{u_j=n} C(\mu_j)$. Since $C(\mu_j)$ has UH-multiplicity $u_j = n$ by Lemma 2.1 (iii), there exists an orthogonal family $\{E'_{n,j,p}\}_{p\varepsilon I_j}$ of abelian projections in W' such that $C_{E'_{n,j,p}} = C(\mu_j)$ for $p \varepsilon I_j$, card. $I_j = n$ and $\sum_{p\varepsilon I_j} E'_{n,j,p} = C(\mu_j)$. By Theorem 66.2 of [3] each $C(\mu_j)$ is countably decomposable in W and hence, by Lemma 2.5 of [8] there exists a vector $x^{(j)}_{n,p} \varepsilon E'_{n,j,p}$ H such that $E'_{n,j,p} = [Wx^{(j)}_{n,p}]$. Consequently, by Theorem 66.2 of [3] $C(\mu_j) = C(\rho(x^{(j)}_{n,p}))$ so that by Theorem 67.3 of [3] there exists a vector $y^{(j)}_{n,p} \varepsilon [Wx^{(j)}_{n,p}]$ such that $\mu_j = \rho(y^{(j)}_{n,p})$ and $[Wy^{(j)}_{n,p}] = [Wx^{(j)}_{n,p}]$. Thus

$$C(\mu_j)\ H = \sum_{p\varepsilon I_j} \oplus [Wy^{(j)}_{n,p}]$$

and by Theorem 60.1 of [3] there exists an isomoprhism $U^{(n)}_j$ from $C(\mu_j)$ H onto $\sum_n \oplus L_2(X,\mathcal{S},\mu_j)$ such that

$$U^{(n)}_j\ E(.)\ C(\mu_j)\ (U^{(n)}_j)^{-1} (f_\ell)_{\ell\varepsilon I_j} = (\chi_{(.)}\ f_\ell)_{\ell\varepsilon I_j}.$$

If $U = \sum_{n\varepsilon M_E} \oplus \sum_{u_j=n} \oplus U_j^{(n)}$, then clearly U is an OTSR of H relative to $E(.)$ with the measure family $(\mu_j)_{j\varepsilon J}$ since $Q_n = \sum_{u_n=n} C(\mu_j)$ and $\sum_{n\varepsilon M_E} Q_n = I$.

(iii) =>(ii). If possible, let $Q = I - \sum_{j\varepsilon J} C(\mu_j) \neq 0$. Then there exists $n \varepsilon M_E$ such that $Q\, Q_n \neq 0$. For $x \varepsilon Q\, Q_n H$, $x \neq 0$, by Theorem 66.2 of [3], $0 \neq C(\rho(x)) = C_{[Wx]} \leq Q\, Q_n$. On the other hand, by (iii) and Theorem 66.5 of [3] we have

$$C(\rho(x)) = \sum_{j\varepsilon J} C(\rho(x))\, C(\mu_j) = 0.$$

This contradiction proves that $Q = 0$.

(iv)$\Rightarrow$ (iii). Trivial.

(v)$\Longrightarrow$(ii). As μ_j has uniform multiplicity $u_j > 0$, by Lemma 2.1 (iii) $C(\mu_j) \neq 0$ for all $j \varepsilon J$. Since $\mu_j \perp \mu_{j'}$, for $j \neq j'$, by Theorem 65.3 of [3] $C(\mu_j)\, C(\mu_j') = 0$. Besides, as $\sum_{n\varepsilon M_E} Q_n = I$, (v) implies that $\sum_{j\varepsilon J} C(\mu_j) = I$.

This completes the proof.

THEOREM 2.5. For a spectral measure $E(.)$ on S with values in projections of H, there exists an OTSR of H relative to $E(.)$.

PROOF. Let $n \varepsilon M_E$. If $\{E_{n,j}\}_{j\varepsilon J_n}$ is a maximal orthogonal family of cyclic projections in WQ_n, then by maximality $\sum_{j\varepsilon J_n} E_{n,j} = Q_n$.

By Proposition 1.3, And Theorem 1.2 and by Lemma 2.5 of [8] there exists an orthogonal family of vectors $\{x_{n,j,p}\}_{p\varepsilon I_j}$ in $E_{n,j}$ H such that

(i) $[Wx_{n,j,p}] = F'_{n,j,p}$ are mutually orthogonal,

(ii) $\sum_{p\varepsilon I_j} F'_{n,j,p} = E_{n,j}$

and

(iii) $C(\rho(x_{n,j,p})) = C_{[Wx_{n,j,p}]} = E_{n,j}$ (vide Theorem 66.2 of [3]), where card. $I_j = n$ and $j \varepsilon J_n$.

Consequently, by Theorem 65.2 of [3] $\rho(x_{n,j,p}) \equiv \rho(x_{n,j,p'})$ for $p,p' \varepsilon I_j$. Thus, let $\mu_{n,j} \equiv \rho(x_{n,j,p})$, $p \varepsilon I_j$. Then by, (iii), $C(\mu_{n,j})\, C(\mu_{n,j'}) = 0$ for $n \neq n'$ or $j \neq j'$. Besides, by Theorem 67.3 of [3] it is clear that $\mu_{n,j}$ has uniform multiplicity n for $j \varepsilon J_n$. Thus $\{\mu_{n,j}\}_{\substack{j\varepsilon J_n \\ n\varepsilon M_E}}$ satisfies the hypothesis and condition (v) of Theorem 2.4 and therefore, is the measure family of an OTSR of H relative to E(.).

NOTE 2.6. An alternate proof of the above theorem can be given as follows:

Let $\{\mu_j\}_{j\varepsilon J}$ be an orthogonal family of non-zero measures in Σ with uniform multiplicity as given in Theorem 49.3 of [3]. If $J_o = \{j \varepsilon J: u(\mu_j) > 0\}$, then as $u(\mu_j)$ is uniform we observe that $C(\mu_j) \neq 0$ for $j \varepsilon J_o$. Then from the discussion on p. 108 of [3] it follows that $\sum_{j\varepsilon J_o} C(\mu_j) = I$ so that, by Theorem 3.4, $\{\mu_j\}_{j\varepsilon J_o}$ is the measure family of an OTSR of H relative to E(.).

3. EQUIVALENCE OF OTSRs. In this section we introduce the concept of equivalence of two OTSRs and give some characterizations of this concept. The principal theorem of this section (Theorem 3.7) obtains a complete set of unitary invariants of spectral measures in terms of the equivalence of OTSRs.

DEFINITION 3.1. Suppose U_i is an OTSR of H_i relative to $E_i(.)$ with the measure family $\{\mu_j^{(i)}\}_{j\varepsilon J_i}$, i=1,2. We say that the OTSRs U_1 and U_2 are equivalent if $u_1(\mu_j^{(1)}) = u_2(\mu_j^{(1)})$, $j \varepsilon J_1$ and $u_1(\mu_j^{(2)}) = u_2(\mu_j^{(2)})$, $j \varepsilon J_2$ and u_1 and u_2 are uniform in each of $\{\mu_j^{(1)}\}_{j\varepsilon J_1}$ and $\{\mu_j^{(2)}\}_{j\varepsilon J_2}$.

The following theorem is immediate from Definition 3.1.

THEOREM 3.2. Any two OTSRs of H relative to E(.) are equivalent.

The following theorem gives some characterizations of the concept of equivalence of OTSRs.

THEOREM 3.3. Suppose U_i is an OTSR of H_i relative to $E_i(.)$ with the measure family $\{\mu_j^{(i)}\}_{j\varepsilon J_i}$, i=1,2. Let $u_2(\mu_j^{(1)})$ and $u_1(\mu_{j'}^{(2)})$ be uniform and positive for $j \varepsilon J_1$ and $j' \varepsilon J_2$. Then the following statements are equivalent.

(i) U_1 and U_2 are equivalent as OTSRs.

(ii) For $j \in J_1$ let $J_{2,j} = \{j' \in J_2 \colon \mu_j^{(1)} \wedge \mu_{j'}^{(2)} \not\equiv 0$ and $u_1(\mu_j^{(1)}) = u_2(\mu_{j'}^{(2)})\}$ and for $j \in J_2$ let $J_{1,j} = \{j' \in J_1 \colon \mu_j^{(2)} \wedge \mu_{j'}^{(1)} \not\equiv 0$ and $u_1(\mu_{j'}^{(1)}) = u_2(\mu_j^{(2)})\}$. Then

(a) $\mu_j^{(1)} \equiv \bigvee_{j' \in J_{2,j}} (\mu_j^{(1)} \wedge \mu_{j'}^{(2)}),\ j \in J_1$

and

(b) $\mu_j^{(2)} \equiv \bigvee_{j' \in J_{1,j}} (\mu_j^{(2)} \wedge \mu_{j'}^{(1)}),\ j \in J_2$

hold.

(iii) $M_{E_1} = M_{E_2}$ and for $n \in M_{E_1}$

(a) $Q_n^{(1)} = \sum_{u_2(\mu_j^{(2)})=n} C_1(\mu_j^{(2)})$

and

(b) $Q_n^{(2)} = \sum_{u_1(\mu_j^{(1)})=n} C_2(\mu_j^{(1)})$.

PROOF.

(i) $\Rightarrow$ (ii) By hypothesis, $u_2(\mu_j^{(1)}) > 0$. Therefore, by Propocition 2.3 (iv) we have

$$\mu_j^{(1)} \equiv \bigvee_{j' \in J_2} (\mu_j^{(1)} \wedge \mu_{j'}^{(2)}).$$

By (i) and the hypothesis that u_1 and u_2 are uniform in $\mu_j^{(1)}$ we have

$$u_1(\mu_j^{(1)}) = u_2(\mu_j^{(1)}) = u_2(\mu_j^{(1)} \wedge \mu_{j'}^{(2)}) = u_2(\mu_{j'}^{(2)})$$

for all those $j \in J_2$ for which $\mu_j^{(1)} \wedge \mu_{j'}^{(2)} \neq 0$. Therefore,

$$\mu_j^{(1)} \equiv \bigvee_{j' \in J_{2,j}} (\mu_j^{(1)} \wedge \mu_{j'}^{(2)}).$$

Similarly, (i) implies (ii) (b).

(ii) => (iii). Let $n \in M_{E_1}$. By Proposition 2.3 (v) the set $J_n^{(1)} = \{j \in J_1 : u_1(\mu_j^{(1)}) = n\}$ is non-empty. Let $j_o \in J_n^{(1)}$. Then by hypothesis (ii) (a) there exists $j' \in J_2$ such that $u_2(\mu_{j'}^{(2)}) = u_1(\mu_{j_o}^{(1)}) = n$. Therefore, again by Proposition 2.3 (v) we have $n \in M_{E_2}$ and hence $M_{E_1} \subset M_{E_2}$. Similarly, (ii) (b) implies that $M_{E_2} \subset M_{E_1}$ and hence $M_{E_1} = M_{E_2}$.

Let $u_1(\mu_j^{(1)}) = n$. Then by (ii) (a) and by Theorem 66.5 of [3] we have

$$C_1(\mu_j^{(1)}) = \sum_{u_2(\mu_{j'}^{(2)})=n} C_1(\mu_j^{(1)})\, C_1(\mu_{j'}^{(2)})$$

since $C_1(\mu_j^{(1)} \wedge \mu_{j'}^{(2)}) = C_1(\mu_j^{(1)})\, C_1(\mu_{j'}^{(2)}) = 0$ whenever $\mu_j^{(1)} \wedge \mu_{j'}^{(2)} = 0$. Then by Proposition 2.3 (vi) we obtain

$$Q_n^{(1)} = \sum_{j\varepsilon J_n^{(1)}} C_1(\mu_j^{(1)}) = (\sum_{j\varepsilon J_n^{(1)}} C_1(\mu_j^{(1)}) (\sum_{u_2(\mu_{j'}^{(2)})=n} C_1(\mu_{j'}^{(2)}))$$

$$= Q_n^{(1)} \sum_{u_2(\mu_{j'}^{(2)})=n} C_1(\mu_{j'}^{(2)}).$$

If $P_n = \sum_{u_2(\mu_{j'}^{(2)})=n} C_1(\mu_{j'}^{(2)})$, then $\{P_n\}_{n\varepsilon M_{E_1}}$ is an orthogonal family of projections and $P_n \geq Q_n^{(1)}$ for each n. Since $\Sigma\, Q_n^{(1)} = I$, we conclude that $P_n = Q_n^{(1)}$ and therefore, (ii) (a) => (iii) (a). Similarly, (ii) (b) => (iii) (b) for $n \varepsilon M_{E_2}$. As $M_{E_1} = M_{E_2}$, (iii) holds.

(iii) => (i) Let $j_o \varepsilon J_1$ and let $u_1(\mu_{j_o}^{(1)}) = n$. Then $n \varepsilon M_{E_1} = M_{E_2}$ so that by (iii) (b)

$$Q_n^{(2)} = \sum_{u_1(\mu_{j'}^{(1)})=n} C_2(\mu_{j'}^{(1)})$$

and therefore, $C_2(\mu_{j'}^{(1)}) \leq Q_n^{(2)}$. Since $u_2(\mu_{j_o}^{(1)}) > 0$ and is uniform, by Lemma 2.1 (iii) $u_2(\mu_{j_o}^{(1)})$ is the same as the UH-multiplicity of $C_2(\mu_{j_o}^{(1)})$. Thus $C_2(\mu_{j_o}^{(1)}) \neq 0$ so that by Proposition 1.3 we conclude that $u_2(\mu_{j_o}^{(1)}) = n$. Thus $u_1(\mu_j^{(1)}) = u_2(\mu_j^{(1)})$ for $j \varepsilon J_1$. Similarly, (iii) (a) and the hypothesis that $M_{E_1} = M_{E_2}$ imply that $u_1(\mu_j^{(2)}) = u_2(\mu_j^{(2)})$, $j \varepsilon J_2$.

NOTE 3.4. Under the hypothesis of the above theorem,

$u_1(\mu_j^{(1)}) = u_2(\mu_j^{(1)})$, $j \in J_1 \Rightarrow$ (ii) (a); and $u_1(\mu_j^{(2)}) = u_2(\mu_j^{(2)})$, $j \in J_2 \Rightarrow$ (ii) (b) of the said theorem; Theorem 3.3 (ii) (a) $\Rightarrow$ $\Rightarrow M_{E_1} \subset M_{E_2}$ and (iii) (a) while Theorem 3.3 (ii) (b) $\Rightarrow M_{E_2} \subset M_{E_1}$ and (iii) (b).

LEMMA 3.5. If $\mu, \nu \in \Sigma$ and $\mu \equiv \nu$, then there exists an iso - morphism U from $L_2(X,\mathcal{S},\mu)$ onto $L_2(X,\mathcal{S}, \nu)$ such that

$$U\chi_{(.)} f = \chi_{(.)} Uf, \quad f \in L_2(X,\mathcal{S},\mu).$$

PROOF. Let $\frac{d\mu}{d\nu} = f_o$ and $\frac{d\nu}{d\mu} = g_o$. Then $f_o \in L_1(X,\mathcal{S},\nu)$ and $g_o \in L_1(X,\mathcal{S},\mu)$, $f_o \geq 0$, $g_o \geq 0$ and $f_o\, g_o = 1$ a.e. Let $Uf = f\, f_o^{1/2}$, $f \in L_2(X,\mathcal{S},\mu)$. Then U is the required isomorphism.

LEMMA 3.6. Let $\{\mu_j\}_{j\in J}$ be an orthogonal family of non-zero members of Σ with $\mu \equiv \bigvee_{j\in J} \mu_j$. Then there exists an isomorphism U from $L_2(X, S,\mu)$ onto $K = \sum_{j\in J} \oplus L_2(X,\mathcal{S},\mu_j)$ such that

$$U\chi_{(.)} f = (\chi_{(.)}(Uf)_j)_{j\in J}\,.$$

PROOF. Clearly, J is countable. Besides, by Theorem 47.2 of [3] there exists a disjoint sequence $\{N_j\}_{j\in J}$ in $\mathcal{S}$ such that $\mu_j = \mu_{N_j}$, $j \in J$, where $\mu_\delta(\sigma) = \mu(\sigma \cap \delta)$. As on p. 17 of [3] we have $\mu \equiv \sum_{j\in J} \mu_{N_j} = \nu$(say). Thus $\mu(X \setminus \bigcup_{j\in J} N_j) = 0$. Due to

Lemma 3.5, without loss of generality, we shall assume $\mu = \nu$. Again by Lemma 3.5 there exists an isomorphism U_j from $L_2(X, S, \mu_j)$ onto $L_2(X, S, \mu_{N_j})$ such that

$$U_j\, \chi_{(.)}\, f_j = \chi_{(.)}\, U_j\, f_j .$$

For $f \in L_2(X, S, \mu)$, let $\tilde{U}f = (f\chi_{N_j})_{j\in J}$. It is easy to verify that $\tilde{U}$ is an isomorphism onto $\sum_{j\in J} \oplus L_2(X, S, \mu_{N_j})$ such that

$$\tilde{U}\chi_{(.)}\, f = (\chi_{(.)}\, f\chi_{N_j})_{j\in J} .$$

If $U_o = \sum_{j\in J} U_j \oplus$ and $U = U_o^{-1}\, \tilde{U}$, then U is the required isomorphism from $L_2(X, S, \mu)$ onto K.

THEOREM 3.7. Suppose U_i is an OTSR of H_i relative to $E_i(.)$ with the measure family $\{\mu_j^{(i)}\}_{j\in J_i}$, i=1,2. Then the following statements are equivalent.

(i) There exists an isomorphism V from H_1 onto H_2 such that $V\, E_1(.)\, V^{-1} = E_2(.)$, which we describe by saying that $E_1(.)$ and $E_2(.)$ are unitarily equivalent.

(ii) $u_1(\mu) = u_2(\mu)$, $\mu \in \Sigma$.

(iii) U_1 and U_2 are equivalent as OTSRs.

PROOF.

(i) => (ii) Suppose V is an isomorphism from H_1 onto H_2 such that $VE_1(.)\,V^{-1} = E_2(.)$. Let $\mu \in \Sigma$. If $y \in C_2(\mu)\,H_2$, then by hypothesis $V^{-1}\,y \in C_1(\mu)\,H_1$. Similarly, if $x \in C_1(\mu)\,H_1$, then $Vx \in C_2(\mu)\,H_2$. Consequently,

$$C_2(\mu)y = V\,C_1(\mu)\,V^{-1}y, \quad y \in H_2.$$

From this it follows that $u_1(\mu) = u_2(\mu)$.

(ii) => (iii). Let $j \in J_1$. For $0 \neq \nu << \mu_j^{(1)}$, by (ii) and by the fact that $u_1(\mu_j^{(1)})$ is uniform we have

$$u_2(\mu_j^{(1)}) = u_1(\mu_j^{(1)}) = u_1(\nu) = u_2(\nu)$$

and hence u_2 is uniform on $\mu_j^{(1)}$. Similarly, u_1 is uniform on $\mu_j^{(2)}$, $j \in J_2$. Thus (iii) holds.

(iii) => (i). By (iii) and Theorem 3.3, $M_{E_1} = M_{E_2}$. Let $n \in M_{E_2}$ and let $J_n^{(1)} = \{j \in J_1 : u_1(\mu_j^{(1)}) = n\}$. Again, by Theorem 3.3 we have

$$\mu_j^{(1)} \equiv \bigvee_{j' \in J_{2,j}} (\mu_j^{(1)} \wedge \mu_{j'}^{(2)}), \quad j \in J_1 .$$

Therefore, by Lemma 3.6 there exists an isomorphism $\tilde{U}_j$ from $L_2(X,\mathcal{S},\mu_j^{(1)})$ onto $\sum_{j' \in J_{2,j}} \oplus\, L_2(X,\mathcal{S},\mu_j^{(1)} \wedge \mu_{j'}^{(2)})$. For $j \in J_1$ and $j' \in J_2$ we observe that $j' \in J_{2,j}$ if and only if $j \in J_{1,j'}$.

For $n \in M_{E_1}$ clearly

$$\sum_{j\varepsilon J_n^{(1)}} \oplus \sum_{j'\varepsilon J_{2,j}} \oplus L_2(X,\mathcal{S},\mu_j^{(1)} \wedge \mu_{j'}^{(2)})$$

is canonically isomorphic to

$$\sum_{j\varepsilon J_n^{(2)}} \oplus \sum_{j\varepsilon J_{1,j'}} \oplus L_2(X,\mathcal{S},\mu_j^{(1)} \wedge \mu_{j'}^{(2)}).$$

Therefore, from the above it follows that there exists a canonical isomorphism U from $\sum_{j\varepsilon J_1} \oplus \sum_{u_1(\mu_j^{(1)})} L_2(X,\mathcal{S},\mu_j^{(1)})$ onto $\sum_{j\varepsilon J_2} \oplus \sum_{u_2(\mu_j^{(2)})} \oplus L_2(X,\mathcal{S},\mu_j^{(2)})$. If $\tilde{U} = U_2^{-1}\, U\, U_1$ then $\tilde{U}$ is an isomorphism from H_1 onto H_2 such that

$$\tilde{U}\, E_1(.)\, \tilde{U}^{-1} = E_2(.).$$

This completes the proof.

NOTE 3.8. The equivalence of (i) and (ii) in the above theorem is directly established in Section 68 of Halmos [3]. Characterization of the unitary equivalence of two arbitrary spectral measures $E_1(.)$ and $E_2(.)$ in terms of the equivalence of the OTSRs U_1 and U_2 in the above theorem is analogous to Theorem 9.9 of [7] or Theorem 8 (iv) of [5], which itself is a generalization of Theorems X.5.12 and XII.3.16 of Dunford and Schwartz [2] to spectral measures with the CGS-property.

4. BOTS AND COBOTS-REPRESENTATIONS. Two new concepts, namely, BOTS-representations and COBOTS-representations of H relative to E(.) are introduced and is shown that H has a BOTS-representation (respy. a COBOTS-representation) relative to E(.) if and only if E(.) has the generalized CGS-property (respy. the CGS-property) in H. Apart from Theorem 3.7 we give some more characterizations of the equivalence of two BOTS-representations and this study sheds more light on the work of Halmos [3] when the spectral measure has the CGS-property or the generalized CGS-property in H.

DEFINITION 4.1. E(.) is said to have the CGS-property (i.e.countable generating set property) in H if there exists a countable set $\mathfrak{X}$ in H such that $[E(\sigma)\, x: \sigma \in \mathcal{S},\, x \in \mathfrak{X}] = H$.

As was shown in [8], E(.) has the CGS-property in H if and only if every Q_n is countably decomposable in W'. This motivates the following

DEFINITION 4.2. A spectral measure E(.) on $\mathcal{S}$ is said to have the generalized CGS-property in H if the projections Q_n are countably decomposable in W.

The following proposition is easily deduced from Theorem 3.1 of [8] and the above definition.

PROPOSITION 4.3. (i) If E(.) has the CGS-property in H, then

E(.) has the generalized CGS-property in H.

(ii) If W is countably decomposable, then E(.) has the generalized CGS-property in H.

(iii) E(.) has the CGS-property in H if and only if W' is countably decomposable.

DEFINITION 4.4. Suppose U is an OTSR of H relative to E(.) with the measure family $\{\mu_j\}_{j\varepsilon J}$. Then we say that U is a bounded OTSR (BOTSR, in abbreviation) of H relative to E(.) if

(i) For each $n \varepsilon M_E$, $\{\mu_j: u(\mu_j) = n\}$ is bounded in Σ.

The OTSR U is called a COBOTSR of H (relative to E(.)) if

(ii) $u(\mu_j) \leq \aleph_0$, $j \varepsilon J$ and U is a BOTSR.

(COBOTSR signifies a BOTSR with the multiplicities of the measures of the representation being countable).

PROPOSITION 4.5. Let U be a COBOTSR of H relative to E(.) with the measure family $\{\mu_j\}_{j\varepsilon J}$. Then:

(i) For $\aleph_0 < n \leq \dim H$, $Q_n = 0$.

(ii) J is countable.

PROOF. (i) is immediate from (ii) of Definition 4.4 and (v) of Proposition 2.3. By (i), M_E is countable and hence (ii) holds by Definition 4.4 (ii).

LEMMA 4.6. Suppose U is an OTSR of H relative to E(.). Then:

(i) If U is a BOTSR of H, then E(.) has the generalized CGS-property in H.

(ii) If U is a COBOTSR of H, then E(.) has the CGS-property in H.

PROOF. Let $\{\mu_j\}_{j\varepsilon J}$ be the measure family of U.

(i) When U is a BOTSR of H, for $n \varepsilon M_E$ the set $\{\mu_j: u(\mu_j)=n\}$ is, by Definition 4.1 (i), bounded in Σ. Since $\overline{\Sigma} = \Sigma/_{\equiv}$ is a boundedly complete lattice, there exists $\nu_n \varepsilon \Sigma$ such that $\nu_n \equiv \vee\{\mu_j: u(\mu_j) = n\}$. Therefore, by Theorem 2.4(v) and by Theorem 66.5 of [3] we have

$$Q_n = \sum_{u(\mu_j)=n} C(\mu_j) = C(\nu_n).$$

Consequently, by Theorem 66.2 of [3] we conclude that Q_n is countably decomposable in W. Hence (i) holds.

(ii) If U is a COBOTSR of H, then by Proposition 4.5 (i) $Q_n=0$ for $n > \aleph_0$. Besides, by (i) Q_n are countably decomposable in W. Then from the equivalence of (ii) and (Viii) of Theorem 3.1 of [8] the result follows.

LEMMA 4.7. If P is a countably decomposable projection in W, then there exists $x \varepsilon PH$ such that $C(\rho(x)) = P$.

PROOF. As W is abelian and P is countably decomposable in W, by Lemma 3.3.12 of [4] P is cyclie. Let $P = [W'x]$. Then by Corollary 2 of Proposition I.1.1 of [1] and by Theorem 66.2 of [3] we have $P = C_{[Wx]} = C(\rho(x))$.

LEMMA 4.8. If E(.) has the generalized CGS-property (respy. CGS-property) in H, then there exists a BOTSR (respy. COBOTSR) U of H relative to E(.).

PROOF. By hypothesis, all Q_n are countably decomposable in W. By Lemma 4.7 there exists $x_n \varepsilon Q_nH$ such that $C(\rho(x_n))=Q_n$, $n \varepsilon M_E$. Let $\mu_{Q_n} = \rho(x_n)$. Then by Theorem 65.1 of [3], $\{\mu_{Q_n}\}_{n \varepsilon M_E}$ is an orthogonal family of non-zero members in Σ. If $0 \neq \nu << \mu_{Q_n}$, then by Theorem 65.3 of [3] there exists a vector $y \varepsilon [Wx_n]$ such that $\rho(y) = \nu$ and hence $C(\nu) \neq 0$. Then by Theorem 66.3 of [3] $C(\nu) \leq C(\mu_{Q_n}) = Q_n$. Therefore, by Proposition 1.3 the H-multiplicity of $C(\nu) = n$ and hence, μ_{Q_n} has uniform multiplicity n. Thus $\{\mu_{Q_n}\}_{n\varepsilon M_E}$ is the measure family of an OTSR U of H relative to E(.) by Theorem 2.4 since $\sum_{n\varepsilon M_E} C(\mu_{Q_n}) = I$ and U is a BOTSR if E(.) has the generalized CGS-property in H. If E(.) has the CGS-property in H, then by Theorem 3.1(ii) of [8], $Q_n = 0$ for $n > \aleph_0$ and hence U is also a COBOTSR.

NOTATION 4.9. Suppose Q_n is countably decomposable. Choose $x_n \varepsilon Q_n H$ such that $C(\rho(x_n)) = Q_n$ (vide Lemma 4.7). If $C(\rho(y)) = Q_n$ then by Theorem 65.2 of [3], $\rho(x_n) \equiv \rho(y)$. We shall denote by μ_{Q_n} the measure $\rho(x_n)$. Then μ_{Q_n} is unique in Σ up to equivalence.

THEOREM 4.10. The spectral measure E(.) has the generalized CGS-property (respy. CGS-property) in H if and only if H has a BOTSR (respy. COBOTSR) relative to E(.).

PROOF. The condition is necessary by Lemma 4.8 and is sufficient by Lemma 4.6.

COROLLARY 4.11. Suppose S is the σ-algebra generated by a countable family of sets. Then H has a COBOTSR relative to E(.) if and only if H is separable. Consequently, H has a COBOTSR relative to a normal operator T, bounded or not, on H (in the sense that it is a COBOTSR relative the resolution of the identity of T) if and only if H is separable.

THEOREM 4.12. Suppose E(.) has the generalized CGS-property in H. If U is an OTSR of H relative to E(.) with the measure family $\{\mu_j\}_{j\varepsilon J}$, then for $n \varepsilon M_E$

$$\mu_{Q_n} \equiv V\{\mu_j: u(\mu_j) = n\} .$$

Consequently, every OTSR of H relative to E(.) is a BOTSR and is, inparticular, a COBOTSR if E(.) has the CGS-property in H. Besides, there exists a BOTSR (respy. COBOTSR) of H relative to E(.) with the measure family $\{\mu_{Q_n}\}_{n\varepsilon M_E}$. All OTSRs of a separable Hilbert space H reltive to any spectral measure E(.) are COBOTSRs.

PROOF. Let $n \in M_E$ and $J_n = \{j \in J: u(\mu_j) = n\}$. From the proof of Lemma 4.8 we have $u(\mu_{Q_n}) = n$ and hence, by Proposition 2.3 (iv)

$$\mu_{Q_n} \equiv \bigvee_{j \varepsilon J} (\mu_{Q_n} \wedge \mu_j).$$

On the other hand, for $j \varepsilon J_n$

$$C(\mu_{Q_n} \wedge \mu_j) = C(\mu_{Q_n})\, C(\mu_j) = Q_n\, C(\mu_j) = C(\mu_j) \qquad (1)$$

by Proposition 2.3 (vi) and by Theorem 66.3 of [3]. Besides, $C(\mu_j) \neq 0$ by Lemma 2.1 (iii). Consequently, $\mu_{Q_n} \wedge \mu_j \neq 0$ for $j \varepsilon J_n$. By Lemma 4.8 $\{\mu_{Q_n}\}_{n \varepsilon M_E}$ is the measure family of a BOTSR of H relative to E(.) so that by Theorem 3.3 (ii) we have

$$\mu_{Q_n} \equiv \bigvee_{j \varepsilon J_n} (\mu_{Q_n} \wedge \mu_j).$$

We claim that $\mu_{Q_n} \wedge \mu_j \equiv \mu_j$, $j \varepsilon J_n$. In fact, on the contrary, there exists $0 \neq \nu_n \ll \mu_j$ such that $\nu_n \perp\!\!\!\perp (\mu_{Q_n} \wedge \mu_j)$ and $\mu_j \equiv \nu_n \vee (\mu_{Q_n} \wedge \mu_j)$. Then by Theorems 66.3 and 66.5 of [3] and by (1) we have

$$C(\mu_j) = C(\nu_n) + C(\mu_{Q_n})\, C(\mu_j) = C(\nu_n) + C(\mu_j)$$

so that $C(\nu_n) = 0$. This shows that $u(\mu_j)$ is not uniform, which is a contradiction. This proves that $\mu_{Q_n} \equiv V\{\mu_j: u(\mu_j) = n\}$.

The rest of the theorem is evident from the first part and the proof of Lemma 4.8.

The following result is immediate from Theorems 4.10 and 4.12.

THEOREM 4.13. E(.) has the generalized CGS-property (respy. CGS-property) in H if and only if every OTSR of H relative to E(.) is a BOTSR (respy. COBOTSR).

NOTE 4.14. Let $J_0 \subset J$ be as in Note 2.6. If E(.) has the CGS-property in H then $0 < u(\mu_j) \leq \aleph_0$ for $j \in J_0$ and $\{\mu_j\}_{j\in J_0}$ in the principal theorem in Section 68 of Halmos [3] is the measure family of a COBOTSR of H by Note 2.6 and Theorem 4.13.

LEMMA 4.15. Let $E_i(.)$ have the generalized CGS-property in H_i and let U_i be a BOTSR of H_i relative to $E_i(.)$ with the measure family $\{\mu_j^{(i)}\}_{j\in J_i}$, i=1,2. Then the following statements are equivalent.

(i) U_1 and U_2 are equivalent as OTSRs.

(ii) $M_{E_1} = M_{E_2}$ and $\mu_{Q_n^{(1)}} \equiv \mu_{Q^{(2)}}$, $n \in M_{E_1}$

(iii) $\mu_{Q_n^{(1)}} \equiv V\{\mu_j^{(2)}: u_2(\mu_j^{(2)}) = n\}$, $n \in M_{E_1}$

and

$$\mu_{Q_n^{(2)}} \equiv \vee \{\mu_j^{(1)}: u_1(\mu_j^{(1)}) = n\}, \; n \in M_{E_2} .$$

PROOF.

(i) => (ii). By Definition 3.1 and Theorem 3.3, $M_{E_1} = M_{E_2}$. Besides, by Theorems 4.12 and 3.3 we have

$$\mu_{Q_n^{(1)}} \equiv \bigvee_{j \in J_n^{(1)}} \mu_j^{(1)} \equiv \bigvee_{j \in J_n^{(1)}} \vee \{\mu_j^{(1)} \wedge \mu_{j'}^{(2)}: j' \in J_{2,j}\}$$

$$\equiv \bigvee_{j,j'} \{\mu_j^{(1)} \wedge \mu_{j'}^{(2)}: u_1(\mu_j^{(1)}) = u_2(\mu_{j'}^{(2)}) = n\}$$

$$\equiv \bigvee_{j' \in J_n^{(2)}} \vee \{\mu_j^{(1)} \wedge \mu_{j'}^{(2)} : j \in J_{1,j'}\}$$

$$\equiv \bigvee_{j' \in J_n^{(2)}} \mu_{j'}^{(2)}$$

$$\equiv \mu_{Q_n^{(2)}}$$

where $J_n^{(i)} = \{j \in J_i: u_i(\mu_j^{(i)}) = n\}$, $i=1,2$ and $J_{1,j'}$, $J_{2,j}$ are as in Theorem 3.3.

(ii) => (iii) by Theorem 4.12 .

(iii) => (i) By (iii) and by Theorem 66.5 of [3] we have

$$Q_n^{(1)} = \sum_{j\varepsilon J_n^{(2)}} C_1(\mu_j^{(2)}) \ , \quad n \ \varepsilon \ M_{E_1} \qquad (1)$$

and

$$Q_n^{(2)} = \sum_{j\varepsilon J_n^{(1)}} C_2(\mu_j^{(1)}), \ n \ \varepsilon \ M_{E_2}. \qquad (2)$$

Besides, by (1) for each $n \ \varepsilon \ M_{E_1}$ there exists $j \ \varepsilon \ J_2$ such that $u_2(\mu_j^{(2)}) = n$ and for each $n \ \varepsilon \ M_{E_2}$ by (2) there exists $j \ \varepsilon \ J_1$ such that $u_1(\mu_j^{(1)}) = n$. Therefore, by Proposition 3.3 (v) we have $M_{E_1} = M_{E_2}$.

Now, for $j \ \varepsilon \ J_2$ let $u_2(\mu_j^{(2)}) = n$. Then $n \ \varepsilon \ M_{E_1}$ and by (iii), $0 \neq \mu_j^{(2)} << \mu_{Q_n^{(1)}}$. Since $\{\mu_{Q_n^{(1)}}\}_{n\varepsilon M_{E_1}}$ is the measure family of a BOTSR of H, $u_1(\mu_{Q_n^{(1)}})$ is uniform and consequently, $u_1(\mu_j^{(2)})$ is uniform and positive. Similarly, $u_2(\mu_j^{(1)})$ is uniform and positive for $j \ \varepsilon \ J_1$. Thus, by (1) and (2) and Theorem 3.3, U_1 and U_2 are equivalent.

The following theorem is immediate from the above lemma and Theorem 3.7.

THEOREM 4.16. Let $E_i(.)$ have the generalized CGS- property in H_i and let U_i be a BOTSR of H_i relative to $E_i(.)$ with the measure family $\{\mu_j^{(i)}\}_{j\varepsilon J_i}$, i=1,2. Then the following statements are equivalent.

(i) $E_1(.)$ and $E_2(.)$ are unitarily equivalent.

(ii) U_1 and U_2 are equivalent as OTSRs.

(iii) $M_{E_1} = M_{E_2}$ and $\mu_{Q_n^{(1)}} \equiv \mu_{Q_n^{(2)}}$, $n \in M_{E_1}$.

(iv) $\mu_{Q_n^{(1)}} \equiv V\{\mu_j^{(2)} : u_2(\mu_j^{(2)}) = n\}$, $n \in M_{E_1}$

and

$$\mu_{Q_n^{(2)}} \equiv V\{\mu_j^{(1)}: u_1(\mu_j^{(1)}) = n\}, \quad n \in M_{E_2}$$

(v) $u_1 = u_2$ on Σ.

NOTE 4.17. The equivalence of (i) and (iii) given in the above theorem generalizes Theorem 27.2 of Plesner and Rohlin [10] for self-adjoint operators on H with certain restrictions.

The following theorem is analogous to Theorem 9.17 of [7] for COBOTSRs.

THEOREM 4.18. Let T be a normal operator (bounded or not) on a separable Hilbert space H with the resolution of the identity E(.). Let $\{\mu_j\}_{j\varepsilon J}$ be an orthogonal family of finite (non-zero) measures on $\mathcal{B}(\mathbb{C})$ with uniform multiplicity $u(\mu_j) > 0$ (relative to E(.)). Let $K = \sum_{j\varepsilon J} \oplus \sum_{u(\mu_j)} \oplus L_2(\mathbb{C}, \mathcal{B}(\mathbb{C}), \mu_j)$. Then an isomorphism U from H onto K is a COBOTSR of H relative E(.) if and only if

(i) $\mu_j(\sigma) = 0$ for $\sigma \in \mathcal{B}(\mathbb{C})$ with $\sigma \cap \sigma(T) = \phi$ and $j \in J$;

(ii) For every scalar Borel measurable function g on $\sigma(T)$ the domain of the normal operator $g(V)$ is given by

$$\mathcal{D}(g(V)) = \{(f_{jk}) \varepsilon K: \sum_j \sum_{j\varepsilon I_j} \int_{\mathbb{C}} |f_{jk}|^2 \, |g|^2 \, d\mu_{jk} < \infty\}$$

and

$$g(V)\,(f_{jk}) = (g\, f_{jk}),\ (f_{jk}) \ \varepsilon\ \mathcal{D}(g(V)),$$

where $V = U T U^{-1}$, $\mu_{jk} = \mu_j$, $k \ \varepsilon \ I_j$ and card. $I_j = u(\mu_j)$.

A similar result holds for an OTSR of H when H is arbitrary.

PROOF. Since $E(\mathbb{C} \setminus \sigma(T)) = 0$ and Lemma 9.16 of [7] is quite general, the argument in the proof of Theorem 9.17 of [7] can be slightly modified to prove the present result. Details are left to the reader.

THEOREM 4.19. Suppose $C(\rho(x))$ has H-multiplicity $n > 0$. Then there exists an orthogonal family $\{\rho(x_j)\}_{j\varepsilon J}$ of measures such that $\rho(x) \equiv \bigvee_{j\varepsilon J} \rho(x_j)$ and $C(\rho(x_j))$ has UH-multiplicity j. Any such orthogonal family $\{\rho(x_j)\}_{j\varepsilon J}$ besides satisfies the following properties:

(i) J is countable.

(ii) $J = \{j \ \varepsilon \ M_E: C(\rho(x))\, Q_j \neq 0\}$.

(iii) $n = \min \{j \ \varepsilon \ J\}$.

(iv) $C(\rho(x_j)) = C(\rho(x))Q_j$ and consequently, $\rho(x_j)$ represents a unique equivalence class in $\bar{\Sigma} = \Sigma_{/\equiv}$.

$\rho(x) \equiv \bigvee_{j\varepsilon J} \rho(x_j)$ as in the above will be called a canonical orthogonal decomposition of $\rho(x)$.

PROOF. Since $C(\rho(x))$ is countably decomposable in W and $\sum_{P\varepsilon M_E} Q_p = I$, we have $J = \{j \varepsilon M_E : C(\rho(x))Q_j \neq 0\}$ is countable. Then for $j \varepsilon J$, by Theorem 66.2 and 58.3 of [3] there exist $x_j \varepsilon C(\rho(x))Q_j H$ such that $C(\rho(x_j)) = C(\rho(x))Q_j$ and consequently, by Proposition 1.3, $C(\rho(x_j))$ has UH-multiplicity j. Besides, by Theorems 65.1 and 66.5 of [3]

$$C(\rho(x)) = \sum_{j\varepsilon J} C(\rho(x))Q_j = \sum_{j\varepsilon J} C(\rho(x_j)) = C(\bigvee_{j\varepsilon J} \rho(x_j))$$

so that by Theorem 65.2 of [3] we have $\rho(x) \equiv \bigvee_{j\varepsilon J} \rho(x_j)$. This proves the first part of the theorem.

Let $\{\rho(y_p)\}_{p\varepsilon J'}$ be an orthogonal family with $\rho(x) \equiv \bigvee_{p\varepsilon J'} \rho(y_p)$, each $C(\rho(y_p))$ having UH-multiplicity p. Clearly, J' is countable. By Proposition 1.3, $C(\rho(y_p)) \leq Q_p$. Then by Theorem 66.5 of [3] $C(\rho(x))\, Q_p \neq 0$. Thus $J' \subset \{j \varepsilon M_E: C(\rho(x))Q_j \neq 0\}$. For $j \varepsilon M_E \setminus J'$,

$$C(\rho(x))Q_j = \sum_{p\varepsilon J'} C(\rho(y_p))Q_j = 0$$

and hence $J' = \{j \in M_E : C(\rho(x))Q_j \neq 0\}$.

By hypothesis,

$$n = \text{H-multiplicity of } C(\rho(x))$$
$$= \min \{\text{H-multiplicity of } C(\rho(x_j)),\ j \in J\}$$
$$= \min \{j : j \in J\}.$$

Finally, as $C(\rho(y_j)) \leq Q_j$ and $C(\rho(x)) = \sum_{j\in J'} C(\rho(y_j))$ we have

$$C(\rho(x))Q_j = C(\rho(y_j))Q_j = C(\rho(y_j))$$

and hence by Theorem 65.2 of [3], $\rho(x_j) \equiv \rho(y_j)$, $j \in J = J'$. This completes the proof.

THEOREM 4.20. Suppose $C(\rho(x)) = I$ and I has UH-multiplicity n. Then there exists an orthogonal family $\{x_j\}_{j\in J}$ in H such that $\rho(x_j) \equiv \rho(x)$, $\sum_{j\in J} \oplus Z(x_j) = H$ and card. $J = n$. Consequently, if $C_1(\rho_1(x)) = I$ and $C_2(\rho_2(y)) = I$ and each of them has UH-multiplicity n relative to the corresponding spectral measure $E_i(.)$, $i=1,2$, then $E_1(.)$ and $E_2(.)$ are unitary equivalent if and only if $\rho_1(x) \equiv \rho_2(y)$.

PROOF. The first part is immediate from the hypothesis, Theorem 65.2 of [3] and the fact that for an abelian projection E'in W' with $C_{E'}$ countably decomposable in W, there exists a vector x such that $E' = [Wx]$ and $C_{E'} = C(\rho(x))$. The last part is immediate from the first and Lemma 1.2 of [7].

THEOREM 4.21. Suppose $C(\rho(x))$ has H-multiplicity n and let $C(\rho(x)) = I$. If $\rho(x) \equiv \bigvee_{j\in J} \rho(x_j)$ is a canonical orthogonal decomposition of $\rho(x)$, then $J = M_E$ and $\rho(x_j) \equiv \mu_{Q_j}$. Consequently, if $C_1(\rho_1(x^{(1)})) = I$ and $C_2(\rho_2(x^{(2)})) = I$, then $E_1(.)$ and $E_2(.)$ are unitarily equivalent if and only if $M_{E_1} = M_{E_2}$ and $\rho_1(x_j^{(1)}) \equiv \rho_2(x_j^{(2)})$ for $j \in J_1$, where $\rho_i(x^{(i)}) \equiv \bigvee_{j\in J_i} \rho_i(x_j^{(i)})$ is a canonical orthogonal decomposition of $\rho_i(x^{(i)})$ with respect to $E_i(.)$, $i=1,2$.

PROOF. Since $C(\rho(x)) = I$, I is countably decomposable en W and hence $E(.)$ has the generalized CGS-property in H. By Theorem 4.19, $J = \{j \in M_E: C(\rho(x))\, Q_j \neq 0\} = M_E$ and $C(\rho(x_j)) = C(\rho(x))\, Q_j = Q_j = C(\mu_{Q_j})$ so that $\rho(x_j) \equiv \mu_{Q_j}$, $j \in J$ by Theorem 65.2 of [3]. Now the last part is immediate from the first and Theorem 4.16.

NOTE 4.22. As is shown in [9], if $E(.)$ is the resolution of the identity of a self-adjoint operator T on H, then $C(\rho(x))$ has H-multiplicity (respy. UH-multiplicity) n if and only if $\rho(x)$ has multiplicity (respy. homogeneous multiplicity) n in the sense of Plesner-Rohlin [10]. Consequently, the above theorems 4.19, 4.20 and 4.21 are the generalizations of the corresponding results of [10] to spectral measures. Also is given the generalization of the principal theorem of [10] on unitary invariants of a general self-adjoint operator to sepectral measures in [9].

5. TOTAL H-MULTIPLICITY OF SPECTRAL MEASURES. In [7] we introduced the notions of OSD-multiplicity, OSR-multiplicity and total multiplicity of a spectral measure E(.) (and of projections commuting with E(.)) and showed that they are all equal. Here we give the concept of total H-multiplicity of a spectral measure E(.) and study its relation with respect to the above multiplicities when E(.) has the CGS-property.

Let us recall some of the definitions from [7].

DEFINITION 5.1. Let $\{x_i\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, be a countable set of non-zero vectors in H such that

(i) $H = \sum_1^N \oplus Z(x_i)$

(ii) $\rho(x_1) \gg \rho(x_2) \gg \ldots.$

Then we say that $H = \sum_1^N \oplus Z(x_i)$ is an ordered spectral decomposition (OSD, in abbreviation) of H relative to E(.). The cardinal number N (which depends solely on E(.)) is called the OSD-multiplicity of E(.). When N is infinite, the OSD-multiplicity is said to be $\aleph_0$. If P is a projection commuting with E(.), then the OSD-multiplicity of P relative to E(.) is defined as that of E(.) P. If T is normal on a separable Hilbert space H with the resolution of the identity E(.), then the OSD-multiplicity of T is defined as that of E(.).

DEFINITION 5.2. Let $\{\mu_i\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, be a countable set of non-zero finite measures on S such that $\mu_1 >> \mu_2 >> \ldots$. Let $K = \sum_1^N \oplus L_2(X, \mathcal{S}, \mu_j)$. If there exists an isomorphism U from H onto K such that

$$U\, E(.)\, U^{-1}(f_j)_1^N = (\chi_{(.)}\, f_j)_1^N$$

then U is called an ordered spectral representation (an OSR, in abbreviation) of H relative to E(.). $\{\mu_i\}_1^N$ is called the measure sequence of U. The cardinal number N (which solely depends on E(.)) is called the OSR-multiplicity of E(.). The OSR-multiplicity of a projection or a normal operator T is given analogously as in Definition 5.1.

NOTE 5.3. If $H = \sum_1^N \oplus Z(x_i)$ is an OSD relative to E(.), then there exists an OSR with the measure sequence $\{\rho(x_i)\}_1^N$ which is said to be the OSR induced by the given OSD; conversely, given an OSR U with the measure sequence $\{\mu_i\}_1^N$, then $H = \sum_1^N \oplus Z(x_i)$ is an OSD, where $x_i = U^{-1}(\delta_{ij})_{j=1}^N$, $\delta_{ij} = 0$, $j \neq i$, $\delta_{ii} = 1$ and this OSD is said to be induced by the OSR U.

Here we adopt the von Neumann definition of ordinal and cardinal numbers, so that each ordinal is identical with the set of all smaller ordinals and a cardinal is an ordinal which cannot be put in one-one correspondence with a smaller ordinal.

DEFINITION 5.4. The total H-multiplicity of E(.) is defined as the supremum of the H-multiplicities of all projections in W (in the well-ordered set of cardinals). If T is normal, then the total H-multiplicity of T is defined as that of its resolution of the identity. If P is a projection commuting with E(.), then the total H-multiplicity of P relative to E(.) is defined as that of P E(.).

The following proposition is immediate from the fact that $\sum_{n \varepsilon M_E} Q_n = I$ and from Theorem 64.2 of [3] and Proposition 1.3.

PROPOSITION 5.5. For a spectral measure E(.) the following statements are equivalent.

(i) The total H-multiplicity of E(.) is $\aleph$.

(ii) $\sup\{n: n \varepsilon M_E\} = \aleph$.

(iii) sup {UH-multiplicity of P, P a projection in W}.

THEOREM 5.6. Suppose E(.) has the CGS-property in H. Then E(.) has the total H-multiplicity N if and only if its OSD-multiplicity is N. Besides, $N \leq \aleph_0$. Consequently, the total H-multiplicity, the total multiplicity, the OSD-multiplicity and the OSR-multiplicity of E(.) are the same.

PROOF. Since E(.) has the CGS-property in H, W' is countably decomposable. Therefore, by Proposition 1.3 $Q_n = 0$ for $n > \aleph_0$. Thus by Proposition 5.5, $N \leq \aleph_0$.

For $n \in M_E$, the projection Q_n is countably decomposable in W' and hence in W. As Q_n has UH-multiplicity n, by Theorem 1.2 there exists an orthogonal family $\{P'_j\}_{j\in J}$ of abelian projections in W' such that card $J = n$, $C_{P'_j} = Q_n$ and $Q_n = \sum_{j\in J} P'_j$. Since P'_j is abelian and $C_{P'_j} = Q_n$ is countably decomposable en W, there exists $x_j \in P'_j H$ such that $P'_j = [Wx_j]$, $j \in J$. If $J = \{1,2,\ldots,n\}$, then $Q_n H = \sum_{j=1}^{n} \oplus Z(x_j)$ and

$$\rho(x_1) \equiv \rho(x_2)\ldots$$

by Theorems 66.2 and 65.2 of [3].

Thus for each $n \in M_E$ there exist vectors $\{x_n^{(j)}\}_{j=1}^{n}$ such that

$$Q_n H = \sum_{j=1}^{n} \oplus Z(x_n^{(j)}) \ , \ \| x_n^{(j)} \| = 1$$

and $\rho(x_n^{(1)}) \equiv \rho(x_n^{(2)}) \equiv \ldots..$

Let $M_E = \{n_1 < n_2 < \ldots.\} \cup \aleph_0$, where $\{n_i\}_1^{\infty} \subset \mathbb{N}$. If $\aleph_0 \notin M_E$ we omit the discussion corresponding to $\aleph_0$. The sequence terminates with n_k if $M_E \cap \mathbb{N}$ is finite and is infinite on the contrary. Thus

$$Q_{n_p} H = \sum_{j=1}^{n_p} \oplus Z(x_{n_p}^{(j)}), \ \rho(x_{n_p}^{(1)}) \equiv \rho(x_{n_p}^{(2)}) \equiv \ldots. \qquad (1)$$

and

$$Q_{\aleph_0} H = \sum_{j=1}^{\infty} \oplus Z(x_{\aleph_0}^{(j)}), \quad \rho(x_{\aleph_0}^{(1)}) \equiv \rho(x_{\aleph_0}^{(2)}) \equiv \ldots . \tag{2}$$

Let $x_{n_p}^{(j)} = 0$ for $j > n_p$, $j \in \mathbb{N}$. We define

$$x_j = \sum_{p=1}^{k'} \frac{1}{n_p} x_{n_p}^{(j)} + x_{\aleph_0}^{(j)}, \quad j \in \mathbb{N}$$

where $k' = \infty$ if M_E is infinite and $k' = k$ if $M_E \cap \mathbb{N} = \{n_1 < n_2 < \ldots < n_k\}$. If $\aleph_0 \notin M_E$ and $k' = k$, then $x_j \neq 0$ only for j in the range $1 \leq j \leq n_k = N$. If $\aleph_0 \in M_E$ on if M_E is infinite, then $x_j \neq 0$ for $j \in \mathbb{N}$. For $\sigma, \delta \in \mathcal{S}$ and $j_1 \neq j_2$ we have

$$\begin{aligned}(E(\sigma)x_{j_1}, E(\delta)x_{j_2}) &= \sum_{p=1}^{k'} \left(\frac{1}{n_p^2} x_{n_p}^{(j_1)}, E(\sigma \cap \delta)x_{n_p}^{(j_2)} \right) \\ &\quad + (x_{\aleph_0}^{(j_1)}, E(\sigma \cap \delta)x_{\aleph_0}^{(j_2)}) \\ &= 0\end{aligned}$$

since $Q_{n_p} Q_{n_{p'}} = 0$ for $p \neq p'$, $Q_{\aleph_0} Q_{n_p} = 0$ and $Z(x_{n_p}^{(j_1)}) \perp Z(x_{n_p}^{(j_2)})$. Consequently, $\{Z(x_j)\}_{j=1}^{N}$ is an orthogonal family of non-zero subspaces of H.

AFFIRMATION. $\rho(x_1) \gg \rho(x_2) \gg \ldots .$

In fact, it suffices to show that $\rho(x_j) \gg \rho(x_{j+1})$. Choose

p_o such that $n_{p_o} < j \leq n_{p_o+1}$, where we take $n_o = 0$. Then $x_{n_p}^{(j)} = x_{n_p}^{(j+1)} = 0$ for $p = 1,2,\ldots,p_o$. Thus

$$x_j = \sum_{p=p_o+1}^{k'} \frac{1}{n_p} \; x_{n_p}^{(j)} + x_{\aleph_o}^{(j)}$$

and

$$x_{j+1} = \sum_{p=p_o+1}^{k'} \frac{1}{n_p} \, x_{n_p}^{(j+1)} + x_{\aleph_o}^{(j+1)}$$

where $x_{n_{p_o+1}}^{(j+1)} = 0$ if $n_{p_o+1} < j+1$, which is the case when $j = n_{p_o+1}$. Suppose $\rho(x_j)\,(\sigma) = 0$. Then

$$\| E(\sigma)x_j\|^2 = \sum_{p=p_o+1}^{k'} \frac{1}{n_p^2} \| E(\sigma)\; x_{n_p}^{(j)}\|^2 + \| E(\sigma)\; x_{\aleph_o}^{(j)}\|^2 = 0$$

so that $\rho(x_{n_p}^{(j)})\,(\sigma) = 0$ for $p_o+1 \leq p \leq k'$ and $\rho(x_{\aleph_o}^{(j)})\,(\sigma) = 0$ (with p finite). From (1) and (2) we have $\rho(x_{n_p}^{(j)}) \equiv \rho(x_{n_p}^{(j+1)})$ if $j+1 \leq n_p$ and $\rho(x_{n_p}^{(j)}) >> \rho(x_{n_p}^{(j+1)}) = 0$ if $j+1 > n_p$; $\rho(x_{\aleph_o}^{(j)}) \equiv$ $\equiv \rho(x_{\aleph_o}^{(j+1)})$ for all j. Thus $\rho(x_{j+1})\,(\sigma) = 0$.

Finally, we assert that $H = \sum_1^N \oplus Z(x_j)$. In fact, on the contrary, suppose $\sum_1^N \oplus Z(x_j) = K \neq H$. Let $y \in H \ominus K$, $y \neq 0$. Then there exists n_{p_o} such that $y_{n_{p_o}} = Q_{n_{p_o}} y \neq 0$ or $Q_{\aleph_o} y = y_{\aleph_o} \neq 0$. It suffices to discuss the case of n_{p_o}. As $Q_{n_{p_o}} Z(x_j) \subset Z(x_j)$, it follows that $Q_{n_{p_o}} y \perp K$. Therefore,

$$0 = (y_{n_{p_o}}, E(\sigma)\, x_j) = (y_{n_{p_o}}, Q_{n_{p_o}} E(\sigma)\, x_j)$$

$$= (y_{n_{p_o}}, E(\sigma) (\tfrac{1}{n_{p_o}} x_{n_{p_o}}^{(j)}))$$

for $\sigma \in$ S and hence $y_{n_{p_o}} \perp Z(x_{n_{p_o}}^{(j)})$ for $j=1,2,\ldots,n_{p_o}$. Consequently, $y_{n_{p_o}} \perp \sum_{j=1}^{n_{p_o}} \oplus Z(x_{n_{p_o}}^{(j)}) = Q_{n_{p_o}} H$ so that $y_{n_{p_o}} = 0$. This contradiction proves that $Q_n\, y = 0$ for $n \in M_E$. Thus $y = 0$ and $K = H$. This shows that

$$H = \sum_{1}^{N} \oplus Z(x_j)$$

is an OSD of H and hence the OSD-multiplicity of E(.) is N.

The last part follows from the first and Theorem 9.20 of [7].

COROLLARY 5.7. If E(.) has the CGS-property in H and P is a projection commuting with E(.) then the total H-multiplicity of P relative to E(.) is the same as its OSD-multiplicity relative to E(.). Consequently, the H-multiplicity of P in W is not greater than its OSD-multiplicity relative to E(.).

The following theorem is an analogue of Theorem 9.22 of [7] for normal operators with total H-multiplicity n, an arbitrary cardinal not greater than the dimension of H.

THEOREM 5.8. Let T be a normal operator on H with its total H-multiplicity n. Then there exists an OTSR U of H onto $K = \sum_{j\varepsilon J} \oplus \sum_{u(\mu_j)} \oplus L_2(\mathbb{C}, \mathcal{B}(\mathbb{C}), \mu_j)$ relative to T with the measure family $(\mu_j)_{j\varepsilon J}$ such that

(i) $\sup\{u(\mu_j): j \varepsilon J\} = n$

and

(ii) $U\ T\ U^{-1} = M_\lambda$, where

$$\mathcal{D}(M_\lambda) = \{(f_{jk}) \varepsilon K: \sum_J \sum_k \int_{\mathbb{C}} |\lambda|^2 \ |f_{jk}(\lambda)|^2 \ d\ \mu_j(\lambda) < \infty\}$$

and

$$M_\lambda(f_{jk}) = (\lambda f_{jk}),\ (f_{jk}) \varepsilon \mathcal{D}(M_\lambda),$$

where $\mathcal{B}(\mathbb{C})$ is the σ-algebra of Borel subsets of $\mathbb{C}$.

We call M_λ the canonical orthogonal representation of T on K.

In the above, U is a COBOTSR of H relative to T if and only if H is separable, in which case $n \leq \aleph_0$.

PROOF.

(i) Holds by Proposition 2.3 (v) and 5.5.

(ii) Let U be an OTSR of H relative to T with the measure family $(\mu_j)_{j \in J}$ as given in Theorem 2.5. Then the argument in the proof of the necessity part of Theorem 9.17 of [7] can be suitably modified to prove (ii). (Note that Lemma 9.16 of [7] is quite general and is applicable here).

The last part is immediate from Corollary 4.11 and (i).

6. COMPARISON OF COBOTSRs WITH OSRs AND OSDs OF [7]. Theorems 2.5 and 9.4 of [7] characterize the CGS-property of E(.) in terms of the existence of an OSD and an OSR of H respectively, while Theorem 4.10 does so in terms of the existence of a COBOTSR of H.

In this section we construct an OSD and an OSR of H relative to E(.) in a canonical way from the given COBOTSR of H, thereby providing an alternate proof of Theorem 2.5 of [7]. Conversely, given an OSR of H (equivalently, an OSD of H relative to E(.)), we construct in a canonical way a COBOTSR of H relative to E(.). Besides, we classify the OSDs (respy. OSRs) into four types and characterize each one of them in terms of the multiplicity set M_E. Thus the results in this section not only explicity give the connection between OSDs (respy. OSRs) and COBOTSRs but also shed more light on the earlier work [7].

NOTATION 6.1. Throughout this section $\{x_j\}_j$ denote the vectors constructed in the proof of Theorem 5.6.

DEFINITION 6.2. Let $H = \sum_1^\infty \oplus Z(w_i)$ be an OSD of H (respy. U an OSR of H with the measure sequence $(\mu_i)_1^\infty$) relative to E(.). Then we say that the OSD (respy. OSR U) is

(i) strongly infinite if there exists an infinite subsequence $\{n_k\}$ such that

$$\rho(w_{n_k}) \underset{\not\equiv}{>>} \rho(w_{n_{k+1}}) \text{ (respy. } \mu_{n_k} \underset{\not\equiv}{>>} \mu_{n_{k+1}})$$

and if there exists $\nu \in \Sigma$ with $C(\nu) \neq 0$ such that

$$\rho(w_j) \underset{\not\equiv}{>>} \nu \text{ (respy. } \mu_j \underset{\not\equiv}{>>} \nu) \text{ for } j \in \mathbb{N};$$

(ii) essentially finite if there exists $n_o \in \mathbb{N}$ such that

$$\rho(w_n) \equiv \rho(w_{n_o}) \quad \text{(respy. } \mu_n \equiv \mu_{n_o})$$

for all $n \geq n_o$;

(iii) semi-finite if there exists an infinite subsequence $\{n_k\}$ such that

$$\rho(w_{n_k}) \underset{\not\equiv}{>>} \rho(w_{n_{k+1}}) \text{ (respy. } \mu_{n_k} \underset{\not\equiv}{>>} \mu_{n_{k+1}})$$

and if there does not exist any $\nu \in \Sigma$ with $C(\nu) \neq 0$ such that

$$\rho(w_j) \underset{\not\equiv}{>>} \nu \quad \text{(respy. } \mu_j \underset{\not\equiv}{>>} \nu) \text{ for all } j \in \mathbb{N}.$$

An OSD $H = \sum_{1}^{N} \oplus Z(w_i)$ (respy. an OSR U with the measure sequence $(\mu_j)_1^N$) of H relative to E(.) is said to be finite if $N \in \mathbb{N}$.

Obviously, the above four types of an OSD (respy. of an OSR) of H are mutually exclusive. Besides, in the light of Theorems 2.9 and 9.8 of [7] it follows that all the OSDs (respy. OSRs) of H relative to E(.) are of the same type.

LEMMA 6.3. Let E(.) have the CGS-property in H. Then:

(i) $\rho(x_j) \equiv V\{\mu_{Q_n} : n \in M_E, n \geq j\}$ for all those j for which $x_j \neq 0$.

(ii) $C(\rho(x_1)) = I$.

(iii) $C(\rho(x_j)) = I - \sum_{\substack{n \in M_E \\ n<j}} Q_n$.

PROOF. Suppose $M_E = \{n_1 < n_2 < \ldots.\} \cup \aleph_0$. The other cases of M_E can be similarly dealt with. With the notations in the proof of Theorem 5.6 we have

$$x_j = \sum_{p=1}^{\infty} \frac{1}{n_p} x_{n_p}^{(j)} + x_{\aleph_0}^{(j)}, \quad j \in \mathbb{N}.$$

Since $Q_{n_p} H = \sum_{j=1}^{n_p} \oplus Z(x_{n_p}^{(j)})$ with $\rho(x_{n_p}^{(1)}) \equiv \rho(x_{n_p}^{(2)}) \equiv \ldots.$, it follows that $C(\rho(x_{n_p}^{(j)})) \equiv C(\rho(x_{n_p}^{(j')})$ for $j,j' = 1,2,\ldots,n_p$.

Consequently, by Theorem 66.2 of [3], $Q_{n_p} H \subset C(\rho(x_{n_p}^{(j)})) =$ $= C_{[Wx_{n_p}^{(j)}]} H \subset Q_{n_p} H$ and hence $C(\rho_{x_{n_p}^{(j)}}) = Q_{n_p}$, $1 \leq j \leq n_p$. Let $w_{n_p} \in Q_{n_p} H$ such that $\rho(w_{n_p}) \equiv \mu_{Q_{n_p}}$. Then by Theorem 65.2 of [3] we conclude that $\rho(x_{n_p}^{(j)}) \equiv \mu_{Q_{n_p}}$, $1 \leq j \leq n_p$. Similarly, $\rho(x_{\aleph_0}^{(j)}) \equiv \mu_{Q_{\aleph_0}}$, $j \in \mathbb{N}$.

As $Q_n Q_m = 0$ for $n \neq m$, by Theorem 65.1 of [3] $\{\rho(x_{n_p}^{(j)})\}_{p=1}^{\infty} \cup \{\rho(x_{\aleph_0}^{(j)})\}$ is an orthogonal family of measures in Σ.

Besides,

$$\rho(x_j) = \sum_{p=1}^{\infty} \frac{1}{n_p^2} \rho(x_{n_p}^{(j)}) + \rho(x_{\aleph_0}^{(j)}), \quad j \in \mathbb{N}.$$

Then by the discussion on p. 79 of [3] for $j \in \mathbb{N}$ with $n_{p_0} < j \leq$ $\leq n_{p_0+1}$ we have

$$\rho(x_j) \equiv \bigvee_{p=1}^{\infty} \rho(x_{n_p}^{(j)}) \vee \rho(x_{\aleph_0}^{(j)})$$

$$\equiv \bigvee_{p_0+1}^{\infty} \rho(x_{n_p}^{(j)}) \vee \rho(x_{\aleph_0}^{(j)})$$

$$\equiv \vee \{\mu_{Q_n} : n \in M_E,\ n \geq j\} \tag{1}$$

since $x_{n_p}^{(j)} = 0$ for $p = 1,2,\ldots, p_0$. This proves (i).

By Theorem 66.5 of [3] from (1) we have

$$C(\rho(x_1)) = \sum_{n\in M_E} C(\mu_{Q_n}) = \sum_{n\in M_E} Q_n = I$$

and

$$C(\rho(x_j)) = \sum_{\substack{n\in M_E\\ n\geq j}} Q_n = I - \sum_{\substack{n\in M_E\\ n<j}} Q_n .$$

Thus (ii) and (iii) hold.

DEFINITION 6.4. Let E(.) have the CGS-property in H. Then we say that E(.) is

(i) strongly infinite if M_E is infinite and $\aleph_0 \in M_E$;

(ii) essentially finite if M_E is finite and $\aleph_0 \in M_E$;

(iii) semi-finite if M_E is infinite and $\aleph_0 \notin M_E$, and

(iv) finite if M_E is finite and $\aleph_0 \notin M_E$.

LEMMA 6.5. Let E(.) have the CGS-property in H. Then the following assertions hold:

(i) If $M_E = \{n_1 < n_2 < \dots\} \cup \{\aleph_0\}$, then

(a) $\rho(x_{n_i}) \underset{\neq}{>>} \rho(x_j) \equiv \rho(x_{n_{i+1}})$ for $n_i < j \leq n_{i+1}$, i=0,1,2.

where $n_0 = 0$ and the term corresponding to x_{n_0} is omitted; and

(b) $\rho(x_j) \underset{\neq}{>>} \mu_{Q_{\aleph_0}}$, $j \in \mathbb{N}$.

(ii) If $M_E = \{n_1 < n_2 < \ldots < n_k\} \cup \aleph_0$, then (i) (a) holds for $i = 0,1,\ldots,k-1$ and

(c) $\rho(x_j) \equiv \mu_{Q_{\aleph_0}}$, $j > n_k$.

(iii) If $M_E = \{n_1 < n_2 < \ldots\}$, then (i) (a) holds and there does not exist $\nu \in \Sigma$ with $C(\nu) \neq 0$ such that

$$\rho(x_j) \underset{\equiv}{>>} \nu \quad \text{for all} \quad j \in \mathbb{N}.$$

(iv) If $M_E = \{n_1 < n_2 < \ldots < n_k\}$, then (i) (a) holds for $i=0,1,2,\ldots,k-1$.

(v) Consequently, if $E(.)$ is strongly infinite (respy. essentially finite, semi-finite, finite) then the same is true for every OSD of H relative to $E(.)$.

PROOF.

(i) Let $M_E = \{n_1 < n_2 < \ldots\} \cup \{\aleph_0\}$. As in the proof of Theorem 5.6 the x_j are non-zero vectors for all $j \in \mathbb{N}$. Since $\mu_{Q_{\aleph_0}} \neq 0$, it follows from Lemma 6.3 (i) that $\rho(x_j) \underset{\neq}{>>} \mu_{Q_{\aleph_0}}$. Besides, for $n_p < j \leq n_{p+1}$, from (i) of Lemma 6.3 we have

$$\rho(x_{n_p}) \underset{\neq}{>>} \rho(x_j) \equiv \rho(x_{n_{p+1}}).$$

Thus (i) holds.

(ii) Suppose $M_E = \{n_1 < n_2 < \ldots < n_k\} \cup \{\aleph_0\}$. Obviously, $x_j = x^{(j)}_{\aleph_0}$ for $j > n_k$ and hence $\rho(x_j) \overset{o}{\equiv} \rho(x^{(j)}_{\aleph_0}) \equiv \mu_{Q_{\aleph_0}}$ for $j > n_k$. As in the case of (i), by Lemma 6.3 (i) we have

$$\rho(x_{n_i}) \underset{\neq}{>>} \rho(x_j) \equiv \rho(x_{n_{i+1}})$$

for $n_i < j \leq n_{i+1}$, $i=0,1,.,\ k-1$.

(iii) If $M_E = \{n_1 < n_2 < \ldots\}$, clearly the arguments in the proof of (i) (a) hold here verbatim. If there exist $\nu \in \Sigma$ with $C(\nu) \neq 0$ such that $\rho(x_j) \underset{\neq}{>>} \nu$ for all $j \in \mathbb{N}$, then by Theorem 66.3 of [3] and by Lemma 6.3 (iii) we have

$$C(\rho(x_j)) = (I - \sum_{\substack{n<j \\ n\in M_E}} Q_n) \geq C(\nu) .$$

Consequently,

$$0 = (I - \sum_{n\in M_E} Q_n) = \bigwedge_{j\in\mathbb{N}} (I - \sum_{\substack{n<j \\ n\in M_E}} Q_n) \geq C(\nu) \neq 0.$$

Thus this contradiction proves (iii).

(iv) The proof of (iv) is similar to that of (i) (a) and we note that $x_j = 0$ for $j > n_k$.

(v) This is immediate from Definitions 6.2 and 6.4, Theorem 2.9 of [7] and the earlier parts of the lemma.

THEOREM 6.6. Let E(.) have the CGS-property in H. Then:

(i) The total H-multiplicity of E(.) is $\aleph_0$ if and only if $x_j \neq 0$ for $j \in \mathbb{N}$.

(ii) $H = \sum_1^{\infty} \oplus Z(x_j)$ is an OSD of H relative to E(.) if the total H-multiplicity of E(.) is $\aleph_0$. Besides, it is a strongly infinite (respy. essentially finite, semi-finite) OSD if and only if E(9) is strongly infinite (respy. essentially finite, semi-finite).

(iii) $H = \sum_1^{N} \oplus Z(x_j)$, $N \in \mathbb{N}$ is a finite OSD of H if and only if E(.) is finite. Then N is the total H-multiplicity of E(.).

(iv) $M_E \cap \mathbb{N} = \{i \in \mathbb{N} : \rho(x_i) \underset{\neq}{>>} \rho(x_{i+1})\}$, where $\rho(x_{i+1}) = 0$ if i is the total H-multiplicity of E(.) and $\aleph_0 \in M_E$ if and only if $\sum_{n \in M_E \cap \mathbb{N}} Q_n \neq I$.

(v) If V is an OSR of H relative to E(.), then V is strongly infinite (respy. essentially finite, semi-finite, finite) if and only if the same is true for E(.).

(vi) Suppose U is a COBOTSR of H relative to E(.) with the measure family $\{\mu_j\}_{j \in J}$. Then there exists an OSR V of H relative to E(.) with the measure sequence $\{\nu_i\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, where

$$\nu_i \equiv V\{\mu_j : u(\mu_j) \geq i\}$$

for all those $i \in \mathbb{N}$ for which there exists some μ_j with $u(\mu_j) \geq i$. Such an OSR V is called the OSR induced by U and is unique upto equivalence (vide [7]). The OSD induced by such V is also called the OSD induced by U and is unique upto equivalence as OSDs (vide [7]).

(vii) If V_1 and V_2 are the OSRs induced by the COBOTSRs U_1 and U_2 of H relative to $E(.)$, then V_1 and V_2 are equivalent as OSRs (vide [7]). Similar result holds for the OSDs induced by U_1 and U_2.

PROOF.

(i) If M_E is infinite or if $\aleph_0 \in M_E$, from the definition of x_j in the proof of Theorem 5.6 it is clear that $x_j \neq 0$ for $j \in \mathbb{N}$. Conversely, suppose $x_j \neq 0$ for $j \in \mathbb{N}$. If the total H-multiplicity of $E(.)$ is finite, then M_E is of the form $M_E = \{n_1 < n_2 < \ldots < n_k\} \subsetneq \mathbb{N}$. Since $x^{(j)}_{n_p} = 0$ for $j > n_p$, it follows that $x_j = 0$ for $j > n_p$. This contradiction proves that the condition is also sufficient.

(ii) From the last part of the proof of Theorem 5.6 and (i) it follows that $H = \sum_1^\infty \oplus Z(x_i)$ is an OSD if the total H-multiplicity of $E(.)$ is $\aleph_0$.

(a) By Lemma 6.5 (v) the OSD is strongly infinite if $E(.)$ is strongly infinite. Conversely, suppose the OSD is strongly infinite. By (i), the total H-multiplicity of $E(.)$ is $\aleph_0$. If $\aleph_0 \notin M_E$, then $M_E \cap \mathbb{N}$ is infinite and by Lemma 6.5 (iii) the OSD is not strongly infinite. If $M_E \cap \mathbb{N}$ is finite and $\aleph_0 \in M_E$, then by Lemma 6.5 (ii) again the OSD fails to be strongly infinite. Therefore, we conclude that $M_E \cap \mathbb{N}$ is infinite and $\aleph_0 \in M_E$. That is, E(.) is strongly infinite.

(b) If E(.) is essentially finite, then by Lemma 6.5 (v) the OSD is essentially finite. Conversely, let the OSD be essentially finite. Let $n_0 \in \mathbb{N}$ such that $\rho(x_i) \equiv \rho(x_{n_0})$ for $i \geq n_0$. Clearly, M_E can not be infinite by (i) and (iii) of Lemma 6.5. On the other hand, by (i) the total H-multiplicity of E(.) is $\aleph_0$ and hence $\aleph_0 \in M_E$. Thus E(.) is essentially finite.

(c) If E(.) is semi-finite, then by Lemma 6.5 (v) the OSD is semi-finite. Conversely, if the OSD is semi-finite, then by (i) and by the above cases (a) and (b) we conclude that E(.) is semi-finite.

(iii) The first part is immediate from Lemma 6.5 (v) and from (ii). The last part follows from Theorem 5.6.

(iv) If $i \in M_E \cap \mathbb{N}$, then by Lemma 6.5 we have $\rho(x_i) \underset{\neq}{\gg} \rho(x_{i+1})$ where $\rho(x_{i+1}) = 0$ if i is the total H-multiplicity of E(.). Conversely, if $\rho(x_i) \underset{\neq}{\gg} \rho(x_{i+1})$, then by Lemma 6.3 (i) we conclude that $i \in M_E$. The last part is due to the fact that $\sum_{n \in M_E} Q_n = I$ and that the total H-multiplicity of E(.) is less than or equal to $\aleph_0$.

(v) This follows from (ii), (iii) and Theorems 2.9 and 9.6 of [7].

(vi) This is immediate from Lemma 6.3 (i) and Theorem 4.12 and from the fact that $H = \Sigma \oplus Z(x_i)$ is an OSD relative to E(.) (vide Theorem 9.6 of [7] and the last part of the proof of Theorem 5.6).

(vii) Let $\{\mu(i)_j\}_{j \in J_i}$ be the measure family of U_i, i=1,2. Let N be the total H-multiplicity of E(.). Let $\{\nu_j^{(i)}\}_{j=1}^{N}$ be the measure sequence of V_i, i=1,2. Then

$$\begin{aligned}
\nu_i^{(1)} &\equiv \bigvee\{\mu_j^{(1)} : u(\mu_j^{(1)}) \geq i\} \\
&\equiv \bigvee\{\mu_{Q_n} : n \in M_E,\ n \geq i\} \\
&\equiv \bigvee\{\mu_j^{(2)} : u(\mu_j^{(2)}) \geq i\} \\
&\equiv \nu_i^{(2)}
\end{aligned}$$

by (vi) and Theorem 4.12. Hence V_1 and V_2 are equivalent as OSRs. The last part is immediate from the first.

This completes the proof.

NOTE 6.7. Theorems 4.12 and 6.6 provide an alternate proof of Theorem 2.5 of [7] . Besides, the present study in Theorem 6.6 gives more details about the OSDs and OSRs in terms of the behaviowr of M_E. Also, the above theorem brings out clearly the connections between OTSRs and OSDs (or OSRs) when E(.) has the CGS-property in H.

The next theorem, among other things, includes the construction of a COBOTSR of H in a canonical way from the given OSD (or OSR) of H.

THEOREM 6.8. Let $H = \sum_{1}^{N} \oplus Z(w_i)$ be an OSD of H relative to E(.), $N \in \mathbb{N} \cup \{\infty\}$. Then:

(i) For $n \in M_E$,

$$Q_n H = \sum_{\substack{j<n \\ j \in \mathbb{N}}} \oplus Z(Q_n w_j)$$

is an OSD of $Q_n H$ with $\rho(Q_n w_1) \equiv \rho(Q_n w_2) \equiv \ldots\ldots$

Besides, $\mu_{Q_n} \equiv \rho(Q_n w_j)$, $1 \leq j \leq n$, $j \in \mathbb{N}$.

(ii) There exists a COBOTSR V of H relative to E(.) with the measure family $\{\rho(Q_n w_1)\}_{n \in M_E}$.

(iii) M_E is determined by (iv) of Theorem 6.6 if we replace x_i by w_i there.

(iv) If U is an OSR of H relative to E(.) with the measure sequence $\{\mu_j\}_1^N$, let $w_j = U^{-1}(f_{ij})_{i=1}^N$ where $f_{ij} = 0$ for $i \neq j$ and $f_{jj} = 1$. Then (i) and (ii) above hold for these vectors w_j.

The COBOTSR V as in (ii) is called the COBOTSR induced by the OSD $H = \sum_1^N \oplus Z(w_i)$ or by the OSR U.

PROOF. By hypothesis, E(.) has the CGS-property in H. Therefore, by Theorem 4.12, $\{\mu_{Q_n}\}_{n \varepsilon M_E}$ is the measure family of a COBOTSR of H relative E(.). Since any two OSDs of H relative to E(.) are equivalent by Theorem 2.9 of [7] and since $H = \sum_1^N \oplus Z(x_i)$ is an OSD of H by Theorem 5.6, we have by Lemma 6.3 (i)

$$\rho(w_i) \equiv \rho(x_i) \equiv \bigvee\{\mu_{Q_n} : n \varepsilon M_E,\ n \geq i\}$$

for $1 \leq i \leq N$, $i \varepsilon \mathbb{N}$. Thus by Lemma 6.3 (iii), $Z(w_i) \perp Q_j$ for $j \varepsilon M_E$ with $j < i$. Therefore, by Theorems 66.2 and 66.5 of [3] we have

$$C_{[WQ_n w_i]} = Q_n C_{[W w_i]} = Q_n C(\rho(w_i)) = \begin{cases} 0 & \text{if } n < i \\ Q_n & \text{if } n \geq i \end{cases}$$

for $n \varepsilon M_E$. Besides, by Theorem 60.2 of [3] the projections $E_i' = [W Q_n w_i]$, $1 \leq i \leq n$, $(n \varepsilon M_E)$, $i \varepsilon \mathbb{N}$ form an orthogonal family of abelian projections. Since these projections have the same central support Q_n, it follows from Theorem 1.2 and from Theorem 65.2 of [3] that

$$Q_n H = \sum_{\substack{i \leq n \\ i \in \mathbb{N}}} \oplus Z(Q_n w_i)$$

is an OSD of $Q_n H$ with $\rho(Q_n w_1) \equiv \rho(Q_n w_2) \equiv \ldots\ldots$

Consequently, $C(\rho(Q_n w_i)) = Q_n$ so that $\rho(Q_n w_i) \equiv \mu_{Q_n}$ for $1 \leq i \leq n$, $i \in \mathbb{N}$.

(ii) This is immediate from (i) and Theorem 4.12.

(iii) Since any two OSDs of H relative to E(.) are equivalent by Theorem 2.9 of [7], (iii) holds by Theorem 6.6 (v).

(iv) This follows from (i) and (ii) and Theorem 9.6 of [7].

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