A NOTE ON SPECTRAL AND PRESPECTRAL OPERATORS

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ABSTRACT. Results known for spectral and prespectral operators in weakly complete Banach spaces are extended to Banach spaces with Bessaga-Pelczyński property.

1. PRELIMINARIES. In this note by a Banach space we mean a non-zero complex Banach space. Because of varying terminology in literature, we give the following definition.

DEFINITION 1.1. Let Σ be a σ-algebra of subsets of a non-empty set S and X a Banach space. If Γ is a total set of X^*, the dual of X, and $E(.)$ is a Boolean homomorphism of Σ on a Boolean algebra (B.A.) of projections on X, we say that $E(.)$ is a spectral measure of class (Σ, Γ) when $E(\Sigma)$ is norm bounded in the Banach algebra $B(X)$ of all bounded operators in X and when $x^* E(.)x$ is countably additive on Σ, for each $x \in X$ and $x^* \in \Gamma$.

When $E(.)$ is a spectral measure of class (Σ, X^*), by Orlicz-
-Pettis' theorem, $E(.)$ is countably additive on Σ in the
strong operator topology and in this case $E(.)$ is called a
spectral measure on Σ .

An operator $T \in B(X)$ is called a prespectral operator of
class Γ if there is a spectral measure $E(.)$ of class (Σ_p, Γ)
where Γ is total in X^* and Σ_p is the σ-algebra of all Bo-
rel sets in $\mathbb{C}$, such that for each δ in Σ_p, $E(\delta)\, T = T\, E(\delta)$
and the spectrum of T restricted to $E(\delta)X$ is contained in
the closure of δ. Then $E(.)$ is called a resolution of the
identity of class Γ for T.

A prespectral operator T of class X^* is called a spectral
operator. In this case, T has a unique resolution of the
identity.

DEFINITION 1.2. A Banach space X is said to have Bessaga-
-Pelczyński (B-P) property if every weakly un conditionally
convergent series in X is strongly un conditionally conver-
gent.

LEMMA 1.3. Let Ω be a compact T_2 space and X a Banach
space with B-P property. Then every bounded linear trans -
formation $T: C(\Omega) \rightarrow X$ is weakly compact, where $C(\Omega)$ is the
Banach algebra of all complex continuous functions on Ω.

PROOF. By theorem 5 of Bessaga and Pelczyński [1], the hypo-
thesis on X is equivalent to the fact that X does not contain

an isomorphic copy of c_o. Now the lemma follows from Theorem 15, p. 160 of Diestel and Uhl [2].

2. MAIN RESULTS. In what follows X will denote a complex Banach space with B-P property. Since a weakly complete Banach space has B-P property (See [1]) all the results in this section hold for weakly complete Banach spaces also.

THEOREM 2.1. Let A be an algebra of operators in X and·let it be the image under a continuous homomorphism ϕ of the Banach algebra $C(\Omega)$ of all complex functions on a compact Huasdorff space Ω. Then there exists a uniquely determined spectral measure E(.) in X on the Borel sets of Ω, for which

$$\phi(f) = \int_\Omega f(\lambda) E(d\lambda), \quad f \in C(\Omega).$$

PROOF. For a fixed $x \in X$, consider the map $T_x: C(\Omega) \rightarrow X$, defined by $T_x f = \phi(f)x$. Since T_x is a continuous linear transformation and X has B-P property, by Lemma 1.3 we have that T_x is weakly compact. Now following the same argument as in the proof of Theorem 2.5 of Chapter XVII of Dunford and Schwartz [4], the present theorem is established.

We use the terms 'complete' and 'σ-complete' B.A of projec - tions on X in the sense of Definition 3.1 of Chapter XVII of Dunford and Schwartz [4].

LEMMA 2.2. A strongly closed bounded Boolean algebra of projections in X is complete.

PROOF. The proof of Lemma 3.5 of Chapter XVII of Dunford and Schwartz [4] holds here, except that we appeal to Theorem 2.1. above, in stead of Theorem 2.5 of Chapter XVII of [4].

THEOREM 2.3. A bounded B.A. of projections on X is complete if and only if it is strongly closed.

PROOF. The theorem follows from Lemma 2.2. above and from Corollary 3.7 of Chapter XVII of [4].

COROLLARY 2.4. Every bounded B.A. of projections B on X can be embedded in a σ-complete B.A. of projections on X, contained in $\bar{B}^S$ (the closure in the strong operator topology).

PROOF. When B is norm bounded by a finite positive number M, then it is easily verified that $\bar{B}^S$ is a B.A. of projections and that $\| E \| \leq M$, $E \in \bar{B}^S$. Now the corollary follows from Theorem 2.3.

The above corollary leads to the following interesting result about prespectral operators.

THEOREM 2.5. Every prespectral operator T on X is a spectral operator.

PROOF. Let T be a prespectral operator of class Γ on X,

with E(.) a resolution of the identity of class Γ. Then by Definition 1.1 and by Corollary 2.4 there exists a σ-complete B.A. B of projections on X, which contains the range of E(.) on Σ_p. If $\{\delta_n\}$ is a sequence of mutually disjoint sets in Σ_p, then $F_n = \sum\limits_{k=1}^{n} E(\delta_k)$ is in B and $\{F_n\}$ is a non-decreasing sequence of projections in B. Therefore, by Lemma 3.4 of Chapter XVII of [4], for each $x \in X$

$$\lim_n F_n x = (\bigvee_1^\infty F_n)x .$$

In other words,

$$(1) \qquad (\bigvee_1^\infty F_n)x = \lim_{n\to\infty} \sum_1^n E(\delta_k)x, \quad x \in X.$$

But, $x^* E(.)x$ is countably additive for each $x \in X$ and $x^* \in \Gamma$ and therefore from (1), we have

$$(2) \qquad x^*(\bigvee_1^\infty F_n)x = \sum_{k=1}^\infty x^* E(\delta_k)x = x^* E(\bigcup_1^\infty \delta_k)x$$
$$x \in X, \; x^* \in \Gamma.$$

Because Γ is total, (1) and (2) give that

$$(3) \qquad (\bigvee_1^\infty F_n)x = E(\bigcup_1^\infty \delta_k)x, \quad x \in X .$$

From (1) and (3) it follows that E(.) is countably additive in the strong operator topology. Hence T is spectral.

THEOREM 2.6. Let T_1 and T_2 be two commuting spectral operators on X, with $E_1(.)$ and $E_2(.)$ as their respective resolutions of the identity and S_1 and S_2 as their respective scalar parts. If the B.A. of projections B determined by $E_1(.)$ and $E_2(.)$ is bounded, then $T_1 + T_2$ and $T_1 T_2$ are spectral with $S_1 + S_2$ and $S_1 S_2$ as their respective scalar parts and $G_1(.)$ and $G_2(.)$ as their respective resolutions of the identity, where

$$G_1(\delta)x = \int E_2(\delta-\lambda) E_1(d\lambda) x,$$

$$G_2(\delta)x = \int E_2(\delta/\lambda) E_1(d\lambda) x$$

for each $x \in X$ and $\delta \in \Sigma_p$.

PROOF. Let $T_1 = S_1 + N_1$ and $T_2 = S_2 + N_2$ be the canonical de compositions of T_1 and T_2. Then by the generalized Fuglede's theorem (Corollary 3.7, Chapter XV of [4]) $S_1, S_2,$ N_1 and N_2 commute with each other and hence $N_1 + N_2$ and $N_1 S_2 + N_2 S_1 + N_1 N_2$ are quasi-nilpotent. Now the theorem follows from Theorem 4.5 of Chapter XV of [4], Corollary 2.4 above and Theorem 10 of Panchapagesan [5].

REMARK 1. In view of Theorems 2.1 and 2.3 and Corollary 2.4, Theorems 6.10, 6.12, 6.14, 6.21 and 6.22 and note 6.15 of Dowson [3] and Corollaries 2.12, 2.13 and 2.14 and Theorems 3.18, 3.19 and 3.20 and Corollary 3.28 of Chapter XVII

of Dunford and Schwartz [4], which are known to be true for weakly complete Banach spaces, continue to be valid for Banach spaces with B-P property.

It is also possible to extend the theorems of characterisations of spectral operators in weakly complete Banach spaces to operators in Banach spaces with B-P property. To this end we need the following theorem.

THEOREM 2.7. Let $E(.): R \to B(X)$ be a Boolean homomorphism of the algebra of sets R on a B.A. of projections on X and let $E(.)$ be countably additive in the strong operator topology on R, in the sense that whenever $\{\delta_n\}$ is a pairwise disjoint sequence of members of R with $\bigcup_1^\infty \delta_n \epsilon R$, then

$$E(\bigcup_1^\infty \delta_n)x = \sum_1^\infty E(\delta_n)x, \quad x \epsilon X .$$

Then a necessary and sufficient condition that $E(.)$ can be extended uniquely to a spectral measure $\bar{E}(.)$ on $S(R)$, the σ-algebra generated by R, is that the range of $E(.)$ is norm bounded in $B(X)$.

PROOF. If such an extension $\bar{E}(.)$ on $S(R)$ exists, then as in the second part of the proof of Corollary 3.10 of Chapter XVII of [4] we see that $\bar{E}(S(R))$ is a σ-complete B.A. of projections on X and hence is norm bounded by Lemma 3.3 of Chapter XVII of [4]. Conversely, if $E(R)$ is norm bounded in

B(X), then E(R) is a bounded B.A. of projections on X. Hence from Theorem 8 of Panchapagesan [5] and Corollary 2.4 above, the sufficiency of the condition follows.

REMARK 2.   As a consecuence of the above   Theorem, Theorems 3.14, 4.5, 5.2 and 5.15 of Chapter XVI of Dunford and Schwartz [4], which are known true for weakly complete Banach spaces, continue to be valid for Banach spaces with B-P property.

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