ON VECTOR LATTICE-VALUED MEASURES-I

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INTRODUCTION: In [16] Wright attempts to characterise a weakly σ-distributive vector lattice V as one for which each V-valued Baire measure on a compact Hausdorff space is regular. But, there is an error in the proof of this Theorem N in [16] since he tacitly assumes on pp.79-80 [16] that

$$\bigvee_{n=1}^{\infty}\left(\sum_{r=1}^{n} \chi_K \, m0_r\right) = \left(\bigvee_{n=1}^{\infty} \sum_{r=1}^{n} m0_r\right) \chi_K \geq m\left(\bigcup_{1}^{\infty} 0_n\right) \chi_K .$$

However, there is no hypothesis in Theorem N of Wright [16] to demand $\bigvee_{n=1}^{\infty} \sum_{r=1}^{n} m0_r$ to be finite and to lie in $\bigvee[mZ]$ since $\{0_r\}$ need not be pair-wise disjoint. Consequently it remains unsettled whether a V valued Baire measure on a compact Hausdorff space is regular, when V is weakly σ-distributive.

Also Wright attempts to characterise in [18] a weakly (σ,∞)--distributive vector lattice V as one for which each V-valued Baire measure on a compact Hausdorff space can be extended to a regular $\hat{V}$-valued Borel measure. But, again his proof of Lemma 2.1. in [18] is incorrect, as he tacitly assumes at the end of p. 280 [18] that the sequence $\{U_n\}$ is increasing. But this need not happen, though $\{B_n\}$ is an increasing sequence. Thus, it remains unsettled whether a V-valued Baire measure on a compact Hausdorff space can be extended to a regular $\hat{V}$-valued Borel measure, when V is weakly (σ,∞)-distributive.

Following a method different from that of Wrignt we study in [11] the regular Borel and weakly Borel extensions of a $V \cup \{\infty\}$ -valued Baire measure μ_o on a locally compact Hausdorff space T, when V is weakly (σ, ∞)-distributive. There we introduce in [11] growth conditions on μ_o such as ' μ_o being dominated ' or ' μ_o being strongly dominated' and prove that every V-valued dominated Baire measure μ_o on a compact Hausdorff space is regular when V is weakly σ-distributive; every $V \cup \{\infty\}$- valued strongly dominated Baire measure μ_o on a locally compact Hausdorff space T extends uniquely to regular $\hat{V} \cup \{\infty\}$-valued Borel and weakly Borel measures on T, when V is weakly (σ, ∞)-distributive.

In [11], by employing the theory of V-valued contents and the theory of vector lattice valued outer measures we also prove that a V-valued finite Baire measure μ_o on T is dominated if and only if it is strongly dominated, when V is weakly (σ, ∞)-distributive.

Thus the present paper prepares the background for [11], the background being the study of vector lattice-valued outer measures and the carathéodory extension of vector lattice-valued measures . Besides providing a new tool to tackle the problem of regular Borel extension of V-valued Baire measures, the study here is interesting in itself as it unifies the Carathéodory extension procedure in the known special cases of numerical and spectral measures in Banach spaces [10]. Here we assume that V is weakly (σ,∞)-distributive and prove that a bounded V-valued or a $V \cup \{\text{strict.}\infty\}$-valued (See Definitions 4.3 and 4.9) measure μ on a ring $\mathcal{R}$ of sets admits the Carathéodory extension.

However, in this connection we may recall here the work of Fremlin [3], Mathes [8] and Wright [16,17] in the extension problem of V-valued measures. They have proved that the weaker hypothesis of weak σ-distributivity of the vector lattice V would itself ensure the solution of the extension problem of V-valued measures. But the extended measure in their work is not required to be defined and countably subadditive on $H(\mathcal{R})$, the hereditary σ-ring generated by $\mathcal{R}$, as is required in the Carathéodory extension procedure. Thus it is not known whether the Carathéodory extension is still possible when V is just weakly σ-distributive and not weakly (σ,∞)-distributive.

In §1, we give the basic definitons and known results from [14,15,16,18], which are needed in the sequel. In §2, the notion of an outer measure is extended to vector lattice-valued set functions and some basic results of such outer measures are obtaind.

As a preliminary to the Carathéodory extension procedure of vector lattice-valued measures, we develop in §3 the theory of induced vector lattice-valued inner measures. In §4, we introduce the notion of $V \cup \{\text{strict.}\infty\}$-valued measures. Any bounded V-valued measure is $V \cup \{\text{strict.}\infty\}$-valued. An extended real valued measure is $V \cup \{\text{strict.}\infty\}$-valued, when $V = \mathbb{R}$. We prove that when V is weakly

(σ, ∞)-distributive, every $V \cup \{\text{strict.}\infty\}$-valued measure on a ring $\mathcal{R}$ admits the Carathéodory extension. The classical Carathéodory extension of extended real valued measures follows as a particular case of this theorem. §5 is devoted to the study of measurable covers and outer regularity of $V \cup \{\text{strict.}\infty\}$-valued measures to obtain the σ-ring of all μ^*-measurable sets as the completion of $\mathcal{S}(\mathcal{R})$, the σ-ring generated by $\mathcal{R}$. The last section deals with applications to positive operator valued measures in Banach spaces and the Carathéodory extension theorem of [10] for spectral measures in Banach spaces is obtained as a particular case of the general situation studied in §4 .

1. PRELIMINARIES. Throughout this paper V will denote a boundedly σ-complete vector lattice with $\hat{V}$ its Dedekind completion. $V^+ = \{x \in V : x \geq 0\}$. We adjoin an object $+\infty$ not in V and extend the partial ordering and addition operation of V to $V \cup \{\infty\}$ in the obvious way. The supremum of any unbounded collection of elements in V^+ or $\hat{V}^+$ is taken to be ∞ .

DEFINITION 1.1. A $V \cup \{\infty\}$-valued measure is a map $\mu : \mathcal{R} \to V \cup \{\infty\}$, where $\mathcal{R}$ is a ring of subsets of a set T such that

(i) $\mu(E) \geq 0$ for E in $\mathcal{R}$;

(ii) $\mu(\phi) = 0$;

(iii) $\mu(\bigcup_1^\infty E_n) = \bigvee_{n=1}^\infty \sum_{i=1}^n \mu(E_i)$, where $\{E_i\}$ is a sequence of pairwise disjoint sets in $\mathcal{R}$ with $\bigcup_1^\infty E_i \in \mathcal{R}$.

For each positive element h in V, let

$$V[h] = \{b \in v : -rh \leq b \leq rh \text{ for some positive } r \in \mathbb{R}\}$$

where $\mathbb{R}$ denotes the real line.

<u>THEOREM 1.2.</u> (Stone-Krein-Kakutani-Yosida) There exists a compact Hausdorff space S such that $V[h]$ is vector lattice isomophic to $C(S)$, the algebra of all real valued continous functions on S. When V is boundedly complete (σ-complete) then so is $V[h]$, $V[h]$ is a Banach space in the order unit norm, the isomorphism is also isometric and $C(S)$ is a Stone algebrea (σ-Stone algebra) in the sense that S is extremally disconnected (S is totally disconnected with the property that the closure of every countable union of clopen subsets of S is open).

For details one may refer to Kadison [5] and Vulikh [13].

We shall use the terms Stone algebra and σ-Stone algebra in the above sense.

From the results of Wright [18] one can define a weakly (σ, ∞)--distributive vector lattice as below.

<u>DEFINITION 1.3.</u> A σ-Stone algebra $C(S)$ is said to be weakly (σ, ∞)--distributive if and only if each meagre subset of S is nowhere dense. Consequently, a boundedly σ-complete vector lattice V is said to be weakly (σ, ∞)-distributive if for $h > 0$ in V, $V[h]$ is weakly (σ, ∞)-distributive.

<u>PROPOSITION 1.4.</u> A boundedly σ-complete vector lattice V is weakly (σ, ∞)-distributive if and only if $\hat{V}$ is so.

2. Vector lattice-valued outer measures. The notion of an outer measure is extended here to V-valued set functions and some basic results of such V-valued outer measures are obtained.

We refer to Halmos [4] for definitions of (i) ring of sets (ii) σ-ring of sets (iii) hereditary σ-ring of sets (iv) algebra or field of sets (v) $\mathcal{S}(\mathcal{R})$, the σ-ring generated by a ring $\mathcal{R}$ of sets and (vi) $H(\mathcal{R})$, the hereditary σ-ring generated by a ring $\mathcal{R}$ of sets.

DEFINITION 2.1. A set function μ^* on a hereditary σ-ring H is called a $V\cup\{\infty\}$-valued outer measure if it satisfies the following conditions:

(i) its range is contained in $V^+\cup\{\infty\}$;

(ii) it is monotone (i.e. $\mu^*(E) \geq \mu^*(F)$ if $E \supseteq F$, E and $F\varepsilon H$);

(iii) it is countably subadditive (i.e. $\mu^*(\bigcup_1^{\infty} E_n) \leq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(E_i)$, $E_i \varepsilon H$, $i = 1,2\ldots$);

(iv) $\mu^*(\phi) = 0$.

DEFINITION 2.2. Let μ^* be a $V\cup\{\infty\}$-valued outer measure on a hereditary σ-ring H. M_{μ^*} be the collection of all sets E in H for which

$$\mu^*(A) = \mu^*(A\cap E) + \mu^*(A\setminus E)$$

holds for every A in H. The members of M_{μ^*} are called μ^*-measurable sets.

REMARK. A set E in H is in M_{μ^*} if and only if

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$$

for every A in H.

DEFINITION 2.3. A $V \cup \{\infty\}$-valued measure μ on a σ-ring S is said to be complete if whenever $E \in S$ and $\mu(E)=0$, then every subset F of E is in S.

LEMMA 2.4. Let μ^* be a $V \cup \{\infty\}$-valued outer measure on a hereditary σ-ring H. Then M_{μ^*} is a ring and μ^* is finitely additive on M_{μ^*}. Further, for $A \in H$ and $E, F \in M_{\mu^*}$ with $E \cap F = \phi$ we have

$$\mu^*\bigl(A \cap (E \cup F)\bigr) = \mu^*(A \cap E) + \mu^*(A \cap F) . \tag{1}$$

PROOF. The proof is similar to that of Theorem A, §11 of Halmos [4].

LEMMA 2.5. Under the hypothesis of Lemma 2.4, M_{μ^*} is a σ-ring. If $A \in H$ and if $\{E_n\}$ is a disjoint sequence of sets in M_{μ^*} with $\bigcup_1^\infty E_n = E$, then

$$\mu^*(A \cap E) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A \cap E_i) . \tag{2}$$

Consequently, every set of outer measure zero belongs to M_{μ^*} and the set function $\hat{\mu}$ defined for E in M_{μ^*} by $\hat{\mu}(E) = \mu^*(E)$ is a complete $V \cup \{\infty\}$-valued measure on M_{μ^*}.

PROOF. To prove (2) observe that by equation (1) of Lemma 2.4, for each n, we have

$$\mu^*\Bigl(A \cap \bigl(\bigcup_1^n E_i\bigr)\Bigr) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$$

for every A in H and that $\bigcup_{i=1}^{n} E_i \in M_{\mu^*}$. Hence for each n,

$$\mu^*(A) = \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + \mu^*\left(A \setminus \bigcup_{i=1}^{n} E_i\right)$$

$$\geq \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*\left(A \setminus \bigcup_{i=1}^{\infty} E_i\right).$$

Now taking the supremum on both sides of the above inequality as n varies from 1 to ∞, we obtain

$$\mu^*(A) \geq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*\left(A \setminus \bigcup_{i=1}^{\infty} E_i\right)$$

so that

$$\mu^*(A) \geq \mu^*\left(A \bigcup \left(\bigcup_{i}^{\infty} E_i\right)\right) + \mu^*\left(A \setminus \bigcup_{i=1}^{\infty} E_i\right)$$

as μ^* is countably subadditive. Replacing A by $A \cap E$ in the above inequality, we obtain (2). The rest of the lemma follows on similar lines as the numerical analogues in Theorems A, B and C, § 11 of Halmos [4].

3.- The inner measure μ_* induced by a $V \bigcup \{\infty\}$-valued measure μ. In this section as a preliminary to the Carathéodory extension procedure of vector lattice-valued measures, we develop the theory of vector lattice-valued inner measures induced by vector lattice - -valued measures.

We fix the following notations in the sequel. $\mathcal{R}$ is a ring of subsets of a set X, μ is a $V\bigcup\{\infty\}$-valued measure on $\mathcal{R}$ where V is a boundedly σ-complete vector lattice and $\mathcal{R}_\sigma = \{E \subseteq X : E = \bigvee_{1}^{\infty} E_n, E_n \in \mathcal{R}\}$. We say that $\mu(E) < \infty$ or $\mu(E)$ is finite if $\mu(E) \in V$.

LEMMA 3.1. Let μ be a $V \cup \{\infty\}$-valued measure on $\mathcal{R}$. If $\{E_n\}$ is an increasing (decreasing) sequence of sets in $\mathcal{R}$ with $\bigcup_1^\infty E_n \in \mathcal{R}$ ($\bigcap_1^\infty E_n \in \mathcal{R}$ and $\mu(E_n) < \infty$ for some n) then

$$\mu(\bigvee_1^\infty E_n) = \bigvee_1^\infty \mu(E_n) \quad (\mu(\bigcap_1^\infty E_n) = \bigwedge_1^\infty \mu(E_n)).$$

PROOF. The statement for increasing sequence is an easy consequence of the countable additivity of μ. In the decreasing case the result follows from Lemma 3.1. of Wright [14] and Theorem III.2.2 of Vulikh [13].

LEMMA 3.2. Let A be in $\mathcal{R}_\sigma$ with $A = \bigcup_1^\infty E_n = \bigcup_1^\infty F_n$, where $\{E_n\}$ and $\{F_n\}$ are increasing sequences of members of $\mathcal{R}$. Then

$$\bigvee_1^\infty \mu(E_n) = \bigvee_1^\infty \mu(F_n)$$

if μ is a $V \cup \{\infty\}$-valued measure on $\mathcal{R}$.

PROOF. Let $A_{n,k} = E_n \cap F_k$. Then $\{A_{n,k}\}_{n=1}^\infty$ and $\{A_{n,k}\}_{k=1}^\infty$ are increasing sequences of members of $\mathcal{R}$ with their unión F_k and E_n respectively. Hence by Lemma 3.1.

$$\mu(E_n) = \bigvee_{k=1}^\infty \mu(A_{n,k}) \text{ and } \mu(F_k) = \bigvee_{n=1}^\infty \mu(A_{n,k}). \quad (3)$$

Thus

$$\bigvee_{n=1}^\infty \mu(E_n) = \bigvee_{n=1}^\infty \bigvee_{k=1}^\infty \mu(A_{n,k}) . \quad (4)$$

If $\bigvee_{n=1}^{\infty} \mu(E_n) = h \in V$, then the equation (4) implies that $h \geq \mu(A_{n,k})$ for every n,k. Since V is boundedly σ-complete, this implies $\bigvee_{n=1}^{\infty} \mu(A_{n,k})$ exists in V for each k and hence by Theorem I.6.1. of Vulikh [13]

$$\bigvee_{k=1}^{\infty} \bigvee_{n=1}^{\infty} \mu(A_{n,k}) = \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{\infty} \mu(A_{n,k}) .$$

This equality and (3) imply that $\bigvee_{n=1}^{\infty} \mu(E_n) = \bigvee_{k=1}^{\infty} \mu(F_k)$.

If $\bigvee_{n=1}^{\infty} \mu(E_n) = \infty$, then necessarily $\bigvee_{k=1}^{\infty} \mu(F_k) = \infty$ lest the above argument will imply that $\bigvee_{n=1}^{\infty} \mu(E_n) < \infty$, a contradiction. This proves the lemma.

The above lemma permits us to make the following definition.

DEFINITION 3.3. Let A be in $\mathcal{R}_\sigma$ and μ be a $V \cup \{\infty\}$-valued measure on the ring $\mathcal{R}$. Then the inner measure μ_* induced by μ is defined on $\mathcal{R}_\sigma$ by

$$\mu_*(A) = \bigvee_{1}^{\infty} \mu(E_n)$$

where $\{E_n\}$ is an increasing sequence of members of $\mathcal{R}$ with $\bigcup_{1}^{\infty} E_n = A$.

Note that if A is in $\mathcal{R}_\sigma$, by definition of $\mathcal{R}_\sigma$, $A = \bigcup_{1}^{\infty} F_n$, $F_n \in R$. Taking $E_n = \bigcup_{i=1}^{n} F_i$, we see that $A = \bigcup_{1}^{\infty} E_n$ and $\{E_n\}$ is an increasing sequence of members of $\mathcal{R}$. Thus μ_* has $\mathcal{R}_\sigma$ as its domain.

LEMMA 3.4. $\mu_*|\mathcal{R} = \mu$. Further μ_* is finitely additive, monotone and $V^+ \cup \{\infty\}$-valued on $\mathcal{R}_\sigma$.

PROOF. The first statement follows from the definition of μ_* and Lemma 3.2. The monotoneity and the non-negativeness of the range of μ_* are evident. We shall now prove the finite additivity of μ_*.

Let A, B be in $\mathcal{R}_\sigma$ with $A \cap B = \phi$. If $A = \bigcup_1^\infty E_n$ and $B = \bigcup_1^\infty F_n$, where $\{E_n\}$ and $\{F_n\}$ are increasing sequences of members of $\mathcal{R}$, then obviously by Definition 3.3

$$\mu_*(A \cup B) = \bigvee_{n=1}^{\infty} \mu(E_n \cup F_n).$$

If either $\mu_*(A) = \infty$ or $\mu_*(B) = \infty$, then by monotoneity of μ_*, $\mu_*(A \cup B) = \infty = \mu_*(A) + \mu_*(B)$. So let $\mu_*(A)$ and $\mu_*(B)$ be finite. Let $\mu_*(A) + \mu_*(B) = h \in V$.

Then $V[h]$ is boundedly σ-complete and $V[h] \simeq C(S)$, a σ-Stone algebra by Theorem 1.2. Then as μ is additive on $\mathcal{R}$ and as $E_n \cap F_n = \phi$ for $n=1,2,\ldots$,

$$\mu_*(A \cup B) = \bigvee_1^\infty \mu(E_n \cup F_n) = \bigvee_1^\infty (\mu(E_n) + \mu(F_n)) \leq h.$$

Thus $\mu_*(A \cup B) \in V[h] \simeq C(S)$. Let us identify $V[h]$ with $C(S)$. By the dual result of Lemma K of Wright [16] and by the fact that finite union of σ-meagre sets is σ-meagre, there exists a σ-meagre subset M of S such that for $s \in S \setminus M$

$$\begin{aligned}\mu_*(A \cup B)\ (s) &= \sup_n \{\mu(E_n) + \mu(F_n)\}\ (s) \\ &= \lim_n \{\mu(E_n) + \mu(F_n)\}\ (s) \\ &= \lim_n \mu(E_n)(s) + \lim_n \mu(F_n)(s) \\ &= \sup_n \mu(E_n)(s) + \sup_n \mu(F_n)(s) \\ &= \mu_*(A)\ (s) + \mu_*(B)\ (s).\end{aligned}$$

Since $\mu_*(A \cup B)$, $\mu_*(A) + \mu_*(B)$ are in $C(S)$ and differ on a meagre subset of S, $\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$ by Theorem 3.4, Chapter 6 of Kelley [6]. Thus μ_* is additive on $\mathcal{R}_\sigma$ and hence finitely additive on $\mathcal{R}_\sigma$ by finite induction.

<u>LEMMA 3.5.</u> If $\{A_n\}$ is an increasing sequence of members in $\mathcal{R}_\sigma$ with $\bigcup_1^\infty A_n = A$, then $A \in \mathcal{R}_\sigma$ and

$$\mu_*(A) = \bigvee_1^\infty \mu_*(A_n).$$

PROOF. For each n, let $A_n = \bigcup_{j=1}^{\infty} E_{n,j}$, $\{E_{n,j}\}_{j=1}^{\infty}$ being an increasing sequence of members of $\mathcal{R}$. If $B_n = \bigcup_{i,j=1}^{n} E_{i,j}$, then $B_n \subseteq A_n$ and $\{B_n\}$ is an increasing sequence of members of $\mathcal{R}$ with $A = \bigcup_1^{\infty} B_n$. Hence A is in $\mathcal{R}_\sigma$.

Now, by Definition 3.3. and Lemma 3.4, we have

$$\mu_*(A) = \bigvee_1^{\infty} \mu(B_n) \leq \bigvee_1^{\infty} \mu_*(A_n) \leq \mu_*(A) .$$

LEMMA 3.6 μ_* is countably subadditive on $\mathcal{R}_\sigma$.

PROOF. Let $\{A_n\}$ be a sequence of members of $\mathcal{R}_\sigma$ with their union A. Let $A_n = \bigcup_{j=1}^{\infty} E_{n,j}$, where $\{E_{n,j}\}_{j=1}^{\infty}$ is an increasing sequence of members in $\mathcal{R}$. Let $B_n = \bigcup_{i,j=1}^{n} E_{i,j}$. Then $B_n \in \mathcal{R}$ and $\{B_n\}$ is an increasing sequence with

$$\bigcup_1^{\infty} B_n = \bigcup_1^{\infty} A_n = A .$$

Hence A is in $\mathcal{R}_\sigma$.

Then

$$\mu_*(A) = \bigvee_1^{\infty} \mu(B_n) = \bigvee_{n=1}^{\infty} \mu\Big(\bigcup_{i,j=1}^{n} E_{i,j}\Big)$$

$$\leq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu\Big(\bigcup_{j=1}^{n} E_{i,j}\Big)$$

$$\leq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu_*(A_i) .$$

Hence $\mu_*\Big(\bigcup_1^{\infty} A_n\Big) \leq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu_*(A_i)$.

4.- Carathéodory extension of vector lattice-valued measures. In this section we prove mainly that the Carathéodory extension procedure is valid for bounded V-valued and suitably restricted $V \cup \{\infty\}$-valued measures on a ring $\mathcal{R}$ of subsets of X, when V is a weakly (σ, ∞)-distributive vector lattice. The classical Carathéodory extension of extended real valued measures follows as a particular case of this result.

$\mathcal{R}$ will denote a ring of sets, and $\mathcal{S}(\mathcal{R})$ $(H(\mathcal{R}))$ will be the σ-ring (hereditary σ-ring) generated by $\mathcal{R}$ in the sequel.

DEFINITION 4.1. Let μ be a $V \cup \{\infty\}$-valued measure on $\mathcal{R}$ and μ_* on $\mathcal{R}_\sigma$ be the inner measure induced by μ. The set function μ^* on $H(\mathcal{R})$ induced by μ is defined by

$$\mu^*(A) = \bigwedge_{\hat{}} \{\mu^*(F) \; : \; A \subseteq F \in \mathcal{R}_\sigma\}$$

for $A \in H(\mathcal{R})$, where $\hat{V}$ is the Dedekind completion of V.

LEMMA 4.2. If μ is a $V \cup \{\infty\}$-valued measure on $\mathcal{R}$, then μ^* is a $\hat{V}^+ \cup \{\infty\}$-valued set function on $F(\mathcal{R})$. $\mu^* | \mathcal{R}_\sigma = \mu_*$ and μ^* is monotone.

PROOF. The first statement follows from Lemma 3.4 and Definition 4.1. The restriction of μ^* to $\mathcal{R}_\sigma$ coincides with μ_* by monotoneity of μ_*. Monotoneity of μ^* is obvious from Definition 4.1.

DEFINITION 4.3. A V-valued measure μ on $\mathcal{R}$ is said to be bounded if there exists a $h \in V^+$ such that $\mu(E) \leq h$ for every E in $\mathcal{R}$. Then we say μ is bounded by h.

Note that a V-valued measure μ on an algebra $\mathcal{R}$ of subsets of a set X is necessarily bounded, by $\mu(X)$.

<u>LEMMA 4.4.</u> Let μ be a bounded V-valued measure on $\mathcal{R}$, with $\mu(E) \leq h$ for all $E \in \mathcal{R}$. Then $\mu_*(F) \leq h$ for all $F \in \mathcal{R}_\sigma$. Consequently, $\mu^*(A) \leq h$ for all $A \in H(\mathcal{R})$, where μ^* is the set function on $H(\mathcal{R})$ induced by μ.

<u>PROOF.</u> If $F \in \mathcal{R}_\sigma$, then $F = \bigcup_1^\infty E_n$, $\{E_n\}$ an increasing sequence of members of $\mathcal{R}$. Thus $\mu_*(F) = \bigvee_1^\infty \mu(E_n) \leq h$. The last part follows from the firs part and Definition 4.1.

<u>LEMMA 4.5.</u> (Countable subadditivity lemma) If V is a weakly (σ, ∞)-distributive vector lattice and if μ^* is the set function induced by a $V \cup \{\infty\}$-valued measure μ on the ring $\mathcal{R}$ of sets, then

$$\mu^* \left(\bigcup_{i=1}^\infty A_i \right) \leq \bigvee_{n=1}^\infty \sum_{i=1}^n \mu^*(A_i) \qquad (5)$$

when $\mu^*(\bigcup_{i=1}^\infty A_i) \in V$, where $A_i \in H(\mathcal{R})$, $i = 1, 2, \ldots\ldots$

<u>PROOF.</u> If the right hand side of (5) is infinity, trivially inequality (5) holds. Hence let $\bigvee_{n=1}^\infty \sum_{i=1}^n \mu^*(A_i) = h_1 \in \hat{V}$.

By hypothesis that $\mu^*(\bigcup_{i=1}^\infty A_i) \in V$ and by Definition 4.1, there exists an $F_o \in \mathcal{R}_\sigma$ such that $\mu_*(F_o) \in \hat{V}$ and $\bigcup_{i=1}^\infty A_i \subseteq F_o$. Let $\mu_*(F_o) = h_2$. Let $h = h_1 \vee h_2$ in V. Then as V is weakly (σ, ∞)--distributive, $\hat{V}$ and $\hat{V}[h]$ are weakly (σ, ∞)-distributive by Proposition 1.4. and Definition 1.3. Further, by Theorem 1.2, $\hat{V}[h] \simeq C(S)$, a weakly (σ, ∞)-distributive Stone algebra. In the proof we shall hereafter identify $\hat{V}[h]$ with $C(S)$.

From Definition 4.1.,

$$\mu^*(A_i) = \hat{\bigwedge}\{\mu_*(F) : A_i \subsetneq F \in \mathcal{R}_\sigma\}. \qquad (6)$$

For $A_i \subsetneq F \in \mathcal{R}_\sigma$, $F \cap F_o \in \mathcal{R}_\sigma$ and $\mu_*(F \cap F_o) \leq \mu_*(F)$.

$$\text{Hence} \quad \mu^*(A_i) \leq \hat{\bigwedge}\{\mu_*(F \cap F_o) : A_i \subsetneq F \in \mathcal{R}_\sigma\}$$

$$\leq \hat{\bigwedge}\{\mu_*(F) : A_i \subsetneq F \in \mathcal{R}_\sigma\}$$

$$= \mu^*(A_i)$$

by (6). Thus for each i,

$$\mu^*(A_i) = \hat{\bigwedge}\{\mu_*(F \cap F_o) : A_i \subsetneq F \in \mathcal{R}_\sigma\}$$

$$= \hat{\bigwedge}\{\mu_*(F) : A_i \subsetneq F \in \mathcal{R}_\sigma, F \subsetneq F_o\} \qquad (6')$$

so that $\mu^*(A_i)$ is realized as the infimum of a decreasing net of elements in $\hat{V}[h] \simeq C(S)$, for $i=1,2,\ldots\ldots$. Hence by Lemma 1.1. of **Wright** [14], there exists a meagre set $M_i \subsetneq S$ such that

$$\mu^*(A_i)(s) = \inf\{\mu_*(F)(s) : A_i \subsetneq F \in \mathcal{R}_\sigma, F \subsetneq F_o\}$$

for $s \in S \setminus M_i$. This holds for $i = 1,2,\ldots\ldots$ Since countable union of meagre sets is meagre, $M = \bigcup_{i=1}^{\infty} M_i$ is meagre and for $s \in S \setminus M$

and for $i = 1,2,\ldots,$

$$\mu^*(A_i)(s) = \inf\{\mu_*(F)(s) : A_i \subseteq F \in \mathcal{R}_\sigma,\ F \subseteq F_o\}.$$

Since $\hat{V}[h] \simeq C(S)$ is weakly (σ,∞)-distributive, by Definition 1.3 the meagre set M is nowhere dense in S, so that $S \setminus \bar{M}$ is open and dense in S. Let $s_o \in S \setminus \bar{M}$. Then there exists a clopen neighbourhood K of s_o such that $K \subseteq S \setminus \bar{M}$. Then the decreasing net $\{\mu_*(F)\chi_K : A_i \subseteq F \in \mathcal{R}_\sigma,\ F \subseteq F_o\}$ descends pointwise on the compact set S to $\mu_*(A_i)\chi_K$ where χ_K is the characteristic function of K and hence by Dini's Theorem the convergence is uniform. Hence given $\varepsilon > 0$, for each positive integer i, there exists an $F_i \in \mathcal{R}_\sigma$, $F_o \supseteq F_i \supseteq A_i$ so that $\mu_*(F_i) \in C(S)$ such that

$$\mu^*(A_i)\,\chi_K + \varepsilon/2^i \geq \mu_*(F_i)\chi_K \quad .$$

Hence

$$\sum_{i=1}^{n} \mu^*(A_i)\chi_K + \sum_{1}^{n} \varepsilon/2^i \geq \sum_{i=1}^{n} \mu_*(F_i)\chi_K$$

so that

$$\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)\chi_K + \varepsilon \geq \bigvee_{n=1}^{\infty} \left(\sum_{i=1}^{n} \mu^*(A_i)\chi_K + \sum_{i=1}^{n} \varepsilon/2^i \right)$$

$$\geq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu_*(F_i)\chi_K \quad . \qquad (7)$$

As $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) \leq h$, $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) \in C(S)$ and hence by the dual result of Corollary on p.109 of Wright [14],

$$(\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i))\chi_K = \bigvee_{n=1}^{\infty} (\sum_{i=1} \mu^*(A_i)\chi_K) .$$

Using this in inequality (7), we have

$$\{\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)\}\chi_K + \varepsilon \geq \bigvee_{n=1}^{\infty} (\sum_{i=1}^{n} \mu^*(F_i)\chi_K)$$

$$\geq \bigvee_{n=1}^{\infty} (\mu_*(\bigcup_{1}^{n} F_i)\chi_K) \qquad (8)$$

as μ_* is finitely subadditive by Lemma 3.6. Since $\bigcup_{i=1}^{n} F_i \subsetneq F_o$, $\mu_*(\bigcup_{i=1}^{n} F_i) \leq \mu_*(F_o)$ so that $\mu_*(\bigcup_{i=1}^{n} F_i) \in C(S)$. Also by Lemma 3.5. $\mu_*(\bigcup_{i=1}^{\infty} F_i) = \bigvee_{n=1}^{\infty} \mu_*(\bigcup_{i=1}^{n} F_i) \leq \mu_*(F_o)$ and hence $\mu_*(\bigcup_{i=1}^{\infty} F_i) \in C(S)$.

Again by the dual result of Corollary on p.109 of Wright [14]

$$\mu_*(\bigcup_{1}^{\infty} F_i)\chi_K = (\bigvee_{n=1}^{\infty} \mu_*(\bigcup_{i=1}^{n} F_i))\chi_K$$

$$= \bigvee_{n=1}^{\infty} (\mu_*(\bigcup_{i=1}^{n} F_i)\chi_K) .$$

Using this equality in (8), we obtain

$$\left(\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)\right)\chi_K + \varepsilon \geq \mu_*\left(\bigcup_{i=1}^{\infty} F_i\right)\chi_K$$

$$\geq \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right)\chi_K \quad .$$

Since ε is arbitrary, the above inequality implies that

$$\left(\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)\right)\chi_K \geq \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right)\chi_K \ .$$

Specialising this inequality at s_o,

$$\left(\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)\right)(s_o) \geq \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right)(s_o) \ .$$

Since s_o is arbitrary in the dense set $S \setminus \overline{M}$ and since $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)$ and $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right)$ are in $C(S)$, the above inequality implies that

$$\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) \geq \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right)$$

in $C(S)$ and hence in $\hat{V}$.

This completes the proof of the lemma.

<u>DEFINITION 4.6.</u> Of $\{g_n\}$ is a sequence of functions in a Stone algebra $C(S)$, we say that $\bigvee_{n=1}^{\infty} g_n = \infty$ if there exists no $g \in C(S)$

such that $g \geq g_n$ for every n. We say that $\bigvee_{n=1}^{\infty} g_n$ is strictly infinity (strict.∞ in notation) if for each non-null clopen subset K of S $\bigvee_{n=1}^{\infty} (g_n \chi_K) = \infty$, where χ_K denotes the characteristic function of K.

We observe that the supremum of any unbounded sequence of non-negative constant functions in $C(S)$ is strict ∞ in the above sense .

<u>DEFINITION 4.7</u>. Let $C(S)$ be a Stone algebra. A $C(S)\cup\{\infty\}$-valued measure μ on $\mathcal{R}$ of sets is said to be strictly infinity $C(S)$ - -valued ($C(S) \cup$ (strict.∞)-valued in notation) if for each increasing sequence $\{E_n\}$ of sets in $\mathcal{R}$ with $\mu(E_n) \in C(S)$ and $\{\mu(E_n)\}$ not bounded above, $\bigvee_{1}^{\infty} \mu(E_n)$ = strict.∞, in the sense of Definition 4.6.

If μ is an extended real valued measure on $\mathcal{R}$ then observe that μ is strictly infinity valued.

<u>LEMMA 4.8.</u> If μ is a $C(S)\cup\{$strict.$\infty\}$-valued measure on a ring $\mathcal{R}$ of sets, where $C(S)$ is a weakly (σ, ∞)-distributive Stone algebra , the set function μ^* induced by μ is countably subadditive on $H(\mathcal{R})$.

<u>PROOF.</u> Let $\{A_i\}$ be a sequence of sets in $H(\mathcal{R})$, with $A = \bigcup_{1}^{\infty} A_i$. If $\mu^*(A)$ is finite, then by Lemma 4.5,

$$\mu^*(A) \leq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) \text{ .}$$

Thus it suffices to prove that if $\mu^*(A)=\infty$, then $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) = \infty$.

If possible, let $\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) = h_1 \in C(S)$ when $\mu^*(A) = \infty$.

Since

$$\mu^*(A_i) = \bigwedge_{C(S)} \{\mu_*(F) : A_i \subseteq F \in \mathcal{R}_\sigma\}$$

and since $\mu^*(A_i)$ is finite, there exists $G_i \in \mathcal{R}_\sigma$ such that $A_i \subseteq G_i$ and $\mu_*(G_i) \in C(S)$. As discussed in the beginning of the proof of Lemma 4.5, it can be shown that for $i = 1,2,\ldots.$

$$\mu^*(A_i) = \bigwedge_{C(S)} \{\mu_*(F) : A_i \subseteq F \in \mathcal{R}_\sigma , F \subseteq G_i\}$$

so that each member in the infimum collection is in $C(S)$. Hence by Lemma 1.1. of Wright [14] there exists a meagre set $M_i \subseteq S$ such that for $s \in S \setminus M_i$

$$\mu^*(A_i)(s) = \inf\{\mu_*(F)(s) : A_i \subseteq F \in \mathcal{R}_\sigma, F \subseteq G_i\}.$$

Hence $M = \bigcup_{i=1}^{\infty} M_i$ is meagre and for $s \in S \setminus M$ and for $i = 1,2,\ldots.$

$$\mu^*(A_i)(s) = \inf\{\mu_*(F)(s) : A_i \subseteq F \in \mathcal{R}_\sigma, F \subseteq G_i\}.$$

Let $s_o \in S\setminus\overline{M}$. Then there is a clopen neighbourhood K of s_o such that $K \subseteq S\setminus\overline{M}$. By an argument similar to the derivation of inequality (7), given $\varepsilon > 0$, there exist sets $F_i \in \mathcal{R}_\sigma$ with $A_i \subseteq F_i \subseteq G_i$ such that

$$\mu^*(A_i)\chi_K + \frac{\varepsilon}{2^i} \geq \mu_*(F_i)\chi_K$$

and hence

$$\bigvee_{n=1}^{\infty}\left(\sum_{i=1}^{n} \mu^*(A_i)\chi_K\right) + \varepsilon \geq \bigvee_{n=1}^{\infty}\left(\sum_{i=1}^{n} \mu_*(F_i)\right)\chi_K$$

$$\geq \bigvee_{n=1}^{\infty} \left(\mu_*\left(\bigcup_{i=1}^{n} F_i\right)\chi_K\right) .$$

By the same argument as in the derivation of inequality (8) we have

$$\left(\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)\right)\chi_K + \varepsilon \geq \bigvee_{n=1}^{\infty} \left(\mu_*\left(\bigcup_{i=1}^{n} F_i\right)\chi_K\right) . \qquad (9)$$

By Lemma 3.5., $\mu_*\left(\bigcup_1^{\infty} F_n\right) = \bigvee_{n=1}^{\infty} \mu_*\left(\bigcup_{i=1}^{n} F_i\right)$. But $\bigcup_1^{\infty} F_n \supseteq A$ and since $\mu^*(A) = \infty$, by Definition 4.1., $\mu_*\left(\bigcup_1^{\infty} F_n\right) = \infty$.

Let $\bigcup_{i=1}^{n} F_i = L_n$. Then $L_n \in \mathcal{R}_\sigma$ and hence let $L_n = \bigcup_{j=1}^{\infty} E_{n,j}$, where $\{E_{n,j}\}_{j=1}^{\infty}$ is an increasing sequence of members of $\mathcal{R}$.

Then $B_n = \bigcup_{i,j=1}^{n} E_{i,j}$ is an increasing sequence of sets in $\mathcal{R}$,

with $\bigcup_1^\infty B_n = \bigcup_1^\infty L_n = \bigcup_1^\infty F_n$. Thus

$$\infty = \mu_*(\bigcup_1^\infty F_n) = \bigvee_1^\infty \mu_*(B_i)$$

since $\{B_i\}$ is an increasing sequence of sets in $\mathcal{R}$. Since $\mu_* | \mathcal{R} = \mu$, $\bigvee_1^\infty \mu(B_i) = \infty$.

$$\mu(B_n) = \mu(\bigcup_{i,j=1}^{n} E_{i,j}) \leq \mu_*(\bigcup_{i=1}^{n} L_i) .$$

$$= \mu_*(L_n)$$

$$= \mu_*(\bigcup_1^n F_i)$$

$$\leq \sum_1^n \mu_*(F_i) \ \varepsilon\ C(S)$$

since each $\mu_*(F_i) \ \varepsilon\ C(S)$. Thus $\{B_n\}$ is an increasing sequence of sets in $\mathcal{R}$ with $\mu(B_n) \ \varepsilon\ C(S)$ and $\{\mu(B_n)\}_1^\infty$ is not bounded above. This implies by the hypothesis on μ that

$$\bigvee_{n=1}^{\infty} (\mu(B_n)\chi_K) = \infty .$$

But

$$\mu(B_n)\chi_K \leq \mu_*(\bigcup_1^n F_i)\chi_K$$

so that

$$\bigvee_{n=1}^{\infty}(\mu_*(\bigcup_{i=1}^{n} F_i)\chi_K) \geq \bigvee_1^{\infty}(\mu(B_n)\chi_K) = \infty .$$

This contradicts inequality (9) and hence the lemma.

DEFINITION 4.9. Let μ be a $V \cup \{\infty\}$-valued measure on a ring $\mathcal{R}$ of sets. We say that μ is $V \cup \{\text{strict.}\infty\}$-valued on $\mathcal{R}$, if there exists an $h \in V^+$ such that μ is $\hat{V}[h] \cup \{\text{strict.}\infty\}$-valued and that for $E \in \mathcal{R}$ with $\mu(E) \in V$, $\mu(E)$ is in $V[h]$.

REMARK. Any bounded V-valued measure μ on $\mathcal{R}$ is vacuously $V \cup \{\text{strict.}\infty\}$-valued on $\mathcal{R}$.

THEOREM 4.10. (Outer measure theorem) Let V be a weakly (σ, ∞)--distributive vector lattice and let μ be a $V \cup \{\text{strict.}\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set X. Then the set function μ^* on $H(\mathcal{R})$ induced by μ is a $\hat{V} \cup \{\infty\}$-valued outer measure on $H(\mathcal{R})$ and is an extension of μ. If μ is bounded by h, then $\mu^*(A) \leq h$ for all $A \in H(\mathcal{R})$.

PROOF. In view of Lemmas 4.2. 3.4. and 4.4., it suffices to show that μ^* is countably subadditive. But by hypothesis, there exists an $h_1 \in V^+$ such that μ is $\hat{V}[h] \cup \{\text{strict.}\infty\}$-valued and $\hat{V}[h_1]$ is a weakly (σ,∞)-distributive Stone algebra. Then by Lemma 4.8 μ^* is countably subadditive on $H(\mathcal{R})$. Thus if $\{A_i\}$ is a sequence of sets in $H(\mathcal{R})$, then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \bigvee_{\substack{n=1\\ \hat{V}[h_1]}}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) = \bigvee_{\substack{n=1\\ \hat{V}}}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)$$

so that μ^* is a $\hat{V}\cup\{\infty\}$-valued outer measure on $H(\mathcal{R})$.

DEFINITION 4.11. When the set function μ^* induced by μ becomes a $\hat{V}\cup\{\infty\}$-valued outer measure on $H(\mathcal{R})$, μ^* will be called the outer measure induced by μ.

LEMMA 4.12. Let μ be a $V\cup\{\text{strict.}\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set X and V be a weakly (σ,∞)-distributive vector lattice. Then the set function μ^* induced by μ is a $\hat{V}\cup\{\infty\}$-valued outer measure on $H(\mathcal{R})$ and M_{μ^*} is a σ-ring containing $\mathcal{S}(\mathcal{R})$, the σ-ring generated by $\mathcal{R}$.

PROOF. μ^* is a $\hat{V}\cup\{\infty\}$-valued outer measure on $H(\mathcal{R})$ by Theorem 4.10, and M_{μ^*} is a σ-ring by Lemma 2.5. Thus the lemma follows if we prove that $\mathcal{R} \subseteq M_{\mu^*}$.

For this, let $E \in \mathcal{R}$ and $A \in H(\mathcal{R})$. Then

$$\mu^*(A) = \bigwedge_{\hat{V}} \{\mu_*(F) : A \subseteq F \in \mathcal{R}_\sigma\}$$

$$= \bigwedge_{\hat{V}} \{\mu_*\{(F\cap E)\cup(F\setminus E)\},\ A \subseteq F \in \mathcal{R}_\sigma\}. \quad (10)$$

Since $E \in \mathcal{R}$ and $F \in \mathcal{R}_\sigma$, $F \cap E$ and $F \setminus E$ are in $\mathcal{R}_\sigma$ and hence by Lemma 3.4., (10) can be rewritten as

$$\mu^*(A) = \hat{\bigvee} \{\mu_*(F \cap E) + \mu_*(F \setminus E) : A \subseteq F \in \mathcal{R}_\sigma\}$$

$$\geq \hat{\bigvee} \{\mu_*(F \cap E): A \subseteq F \in \mathcal{R}_\sigma\} + \hat{\bigvee} \{\mu_*(F \setminus E): A \subseteq F \in \mathcal{R}_\sigma\}. \qquad (11)$$

But by definition of μ^*,

$$\hat{\bigvee} \{\mu_*(F \cap E): A \subseteq F \in \mathcal{R}_\sigma\} \geq \mu^*(A \cap E)$$

and

$$\hat{\bigvee} \{\mu_*(F \setminus E): A \subseteq F \in \mathcal{R}_\sigma\} \geq \mu^*(A \setminus E).$$

Using these inequalities in (11) we obtain

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E) .$$

Since μ^* is subadditive, the reverse inequality holds and $E \in M_{\mu^*}$. Thus $\mathcal{R} \subseteq M_{\mu^*}$.

This completes the proof of the lemma.

<u>REMARK.</u> The notion of a σ-finite $V \cup \{\infty\}$-valued measure can

be introduced here and it can he shown that in Lemma 4.12, μ^* is σ-finite if μ is σ-finite.

LEMMA 4.13. Let μ be a $V \cup \{\text{strict. } \infty\}$-valued measure on $\mathfrak{R}$ and let V be a weakly (σ, ∞)-distributive vector lattice. Then the outer measure μ^* induced by μ is a complete $\hat{V} \cup \{\infty\}$-valued measure on M_{μ^*} extending μ to M_{μ^*} and $\mathfrak{S}(\mathfrak{R})$. If μ is bounded by h in V then $\bar{\mu} = \mu^* | \mathfrak{S}(\mathfrak{R})$ is further V-valued and bounded by h. The extension $\bar{\mu} = \mu^* | \mathfrak{S}(\mathfrak{R})$ of μ to $\mathfrak{S}(\mathfrak{R})$ is unique when μ is σ-finite.

PROOF. By Theorem 4.10 and Lemma 2.5, μ^* is a complete $\hat{V} \cup \{\infty\}$--valued measure on M_{μ^*}. The uniqueness of the extension $\bar{\mu}$ of μ to $\mathfrak{S}(\mathfrak{R})$, when μ is bounded, or when μ is σ-finite follows from an argument analogous to the numerical case (proof of Theorem A, §13 of Halmos [4]) due to the availability of Lemma 3.1. When μ is bounded by h in V, μ^* and $\bar{\mu}$ are bounded by h, by Theorem 4.10.

Finally, we have to prove that the range of μ^* on $\mathfrak{S}(\mathfrak{R})$ is contained in V if μ is bounded by h. Let $\mathfrak{F}$ be the collection of all sets A in M_{μ^*}, for which $\mu^*(A) \in V$. $\mathfrak{R} \subseteq \mathfrak{F}$. In view of theorem B, §6 of Halmos [4], it suffices to show that $\mathfrak{F}$ is a monotone class. Since μ is bounded by h, $\mu^*(A) \leq h$ for every $A \in M_{\mu^*}$ by Theorem 4.10. Let $\{E_n\}$ be a monotone sequence of sets in $\mathfrak{F}$.

Then as μ^* is a $\hat{V}$-valued measure on M_{μ^*}, by Lemma 3.1.

$$\mu^* \left(\bigcup_1^\infty E_n\right) = \bigvee_1^\infty \mu^*(E_n) \quad \text{(if } \{E_n\} \text{ is increasing)}$$

and

$$\mu^* \left(\bigcap_1^\infty E_n\right) = \bigwedge_1^\infty \mu^*(E_n) \quad \text{(if } \{E_n\} \text{ is decreasing)}.$$

Consequently, as V is boundedly σ-complete and $0 \leq \mu^*(E_n) \leq h \in V$ for all n, we obtain that $\mu^*(\bigcup_1^\infty E_n) \in V$ and $\mu^*(\bigcap_1^\infty E_n) \in V$. Thus $\mathfrak{F}$ is a monotone class and hence μ^* is V-valued on $\mathfrak{S}(\mathcal{R})$.

Thus in the foregoing lemmas of this section we have proved the following theorem.

<u>THEOREM 4.14</u>. (Carathéodory extension theorem) Let μ be a $V \cup \{\text{strict.}\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set X and let V be a weakly (σ, ∞)-distributive vector lattice, with $\hat{V}$ its Dedekind completion. Then μ^*, the set function induced by μ is a $\hat{V} \cup \{\infty\}$-valued outer measure and M_{μ^*} is a σ-ring containing $\mathfrak{S}(\mathcal{R})$. Further, μ^* is a complete $\hat{V} \cup \{\infty\}$-valued measure on M_{μ^*} and the restriction $\bar{\mu}$ of μ^* to $\mathfrak{S}(\mathcal{R})$ is a $\hat{V} \cup \{\infty\}$-valued measure extending μ to $\mathfrak{S}(\mathcal{R})$. If μ is further σ-finite on $\mathcal{R}$, so is μ^* on $H(\mathcal{R})$ and $\bar{\mu} = \mu^* \mid \mathfrak{S}(\mathcal{R})$ is a σ-finite $\hat{V} \cup \{\infty\}$-valued measure extending uniquely μ to $\mathfrak{S}(\mathcal{R})$. If μ is a V-valued measure bounded by h on $\mathcal{R}$, then $\bar{\mu} = \mu^* \mid \mathfrak{S}(\mathcal{R})$ is a V-valued measure extending uniquely μ to $\mathfrak{S}(\mathcal{R})$ and is also bounded by h.

REMARK. Since $\mathbb{R}$ is a weakly (σ, ∞)-distributive Stone algebra $C(S)$, where S is a singleton with discrete topology and since any extended real valued measure is $\mathbb{R} \cup \{\text{strict.}\infty\}$-valued, the above theorem includes the classical carathéodory extension theorem of numerical measures as a particular case.

5.- Completion and outer regularity of vector lattice--valued measures. Throughout this section V will denote a weakly (σ, ∞)-distributive vector lattice.

Let μ be a $V \cup \{\infty\}$-valued measure on a σ-ring $\mathcal{S}$. If $\tilde{\mathcal{S}} = \{E \cup N: E \in \mathcal{S},\ N$ a subset of a set in $\mathcal{S}$ of μ-measure zero$\}$ then $\tilde{\mathcal{S}}$ is a σ-ring. If $\tilde{\mu}$ is defined on $\tilde{\mathcal{S}}$ by $\tilde{\mu}(E \cup N) = \mu(E)$, then $\tilde{\mu}$ is a complete $V \cup \{\infty\}$-valued measure on $\tilde{\mathcal{S}}$. $\tilde{\mu}$ is called the completion of μ and $\tilde{\mathcal{S}}$ is called the completion of $\mathcal{S}$.

In this section we obtain a sufficient condition to obtain M_{μ^*} as $\widetilde{\mathcal{S}(\mathcal{R})}$ where μ is a σ-finite $V \cup \{\text{strict.}\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set X. This result can be compared with the numerical analogue.

DEFINITION 5.1. Let μ be a $V \cup \{\text{strict.}\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set X, where V is a weakly (σ, ∞)--distributive vector lattice and let μ^* be the outer measure on $H(\mathcal{R})$ induced by μ. Then μ is said to be outer regular, if for each set E in $H(\mathcal{R})$, there is a set F in $\mathcal{S}(\mathcal{R})$ such that

(i) $E \subseteq F$;

(ii) if $G \in \mathcal{S}(\mathcal{R})$ with $G \subseteq F \setminus E$, then $\bar{\mu}(G) = 0$

and

(iii) $\mu^*(E) = \bar{\mu}(F)$

where $\bar{\mu} = \mu^* / \mathcal{S}(\mathcal{R})$.

A set F in $\mathcal{S}(\mathcal{R})$ satisfying conditions (i) and (ii) above is called a measurable cover of E .

<u>THEOREM 5.2.</u> Let V be a weakly (σ, ∞)-distributive vector lattice satisfying the countable chain condition. If μ is a σ-finite $V \cup \{\text{strict.}\infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set X, then μ is outer regular.

<u>PROOF.</u> By hypothesis there is an $h \in V$ such that $\mu(E) \in V[h]$ if $E \in \mathcal{R}$ and $\mu(E) < \infty$ and μ is $\hat{V}[h] \cup (\text{strict.}\infty)$-valued. Let $A \in H(\mathcal{R})$.

Case 1. Let $\mu^*(A) < \infty$. Then clearly from the definition of μ^* it follows that $\mu^*(A) \in \hat{V}[h] = V[h]$ since $\hat{V} = V$ as V satisfis the countable chain condition. Also the finiteness of $\mu^*(A)$ implies that there is a set B_0 in $\mathcal{R}_\sigma$ with

$$A \subseteq B_0 , \ \mu_*(B_0) \in V[h].$$

Then as in the derivation of $(6')$ we have

$$\mu^*(A) = \bigwedge_{V[h]}\{\mu_*(B) : A \subseteq B \in \mathcal{R}_\sigma , \ B \subseteq B_0\} .$$

As V satisfies the countable chain condition by Theorem V 1.2.1 of Vulikh [13],

$$\mu^*(A) = \bigwedge_{V[h]} \{ \mu_*(B_n): A \subseteq B_n \in \mathcal{R}_\sigma , B_n \subseteq B_o \}. \quad (12)$$

Let $F_n = \bigcap_{i=1}^{n} B_i$. Then $F_n \supseteq A$, $F_n \in \mathcal{R}_\sigma$, $\{F_n\}$ is a decreasing sequence and $\mu^*(F_n) = \mu_*(F_n) \leq \mu_*(B_o) < \infty$, for each n. Let $F = \bigcap_1^\infty F_n$. Then $F \in \mathcal{S}(\mathcal{R})$ and $F \supseteq A$. By Lemma 3.1 and by the monotoneity of μ^* ,

$$\mu^*(A) \leq \mu^*(F) = \bar{\mu}(F) = \bigwedge_1^\infty \bar{\mu}(F_n) \leq \bigwedge_1^\infty \mu_*(B_n) = \mu^*(A).$$

Thus

$$\mu^*(A) = \bar{\mu}(F), \ A \subseteq F \in \mathcal{S}(\mathcal{R}).$$

Let $G \in \mathcal{S}(\mathcal{R})$ with $G \subseteq F \setminus A$. Then $\bar{\mu}(F) = \mu^*(A) \leq \mu^*(F \setminus G) = \bar{\mu}(F \setminus G) = \bar{\mu}(F) - \bar{\mu}(G)$, and hence $\bar{\mu}(G) = 0$. Thus F is a measurable cover of A.

Case 2. Let $\mu^*(A) = \infty$. Since μ is σ-finite on $\mathcal{R}$, by the remark under Lemma 3.12, μ^* is σ-finite on $H(\mathcal{R})$. Hence there exists a sequence $\{A_i\}$ of sets in $H(\mathcal{R})$ with

$$A \subseteq \bigcup_1^\infty A_i , \ \mu^*(A_i) < \infty \ \text{ for } i = 1,2,\ldots,$$

Therefore by case 1, there exists a measurable cover F_i in $\mathcal{S}(\mathcal{R})$

for each A_i. Let $F = \bigcup_{i=1}^{\infty} F_i$. Then $F \in \mathcal{S}(\mathcal{R})$ and $\mu^*(F) = \infty$. If $G \in \mathcal{S}(\mathcal{R})$ with $G \subseteq F \setminus A$, then

$$G \cap F_i \in \mathcal{S}(\mathcal{R}) \text{ and } G \cap F_i \subseteq F_i \setminus A \subseteq F_i \setminus A_i$$

so that $\bar{\mu}(G \cap F_i) = 0$. Then $\bar{\mu}(G) = \bar{\mu}(\bigcup_1^{\infty} G \cap F_i) \leq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \bar{\mu}(G \cap F_i) = 0$.

Thus F is a measurable cover of A and $\bar{\mu}(F) = \mu^*(A) = \infty$.

This completes the proof of the theorem.

PROPOSITION 5.3. Let V be a weakly (σ, ∞)-distributive vector lattice. If μ is a $V \cup \{\text{strict. } \infty\}$-valued measure on $\mathcal{R}$ with μ^* its induced outer measure on $H(\mathcal{R})$, then the following hold.

(i) If $E \in H(\mathcal{R})$ with F_1 and F_2 as measurable covers then

$$\bar{\mu}(F_1 \triangle F_2) = 0 .$$

(ii) If μ is outer regular, then $\mu^*(E) = \bar{\mu}(F)$ for every measurable cover F of E.

(iii) Further if V satisfies the countable chain condition and μ is σ-finite, then

$$\mu^*(E) = \bar{\mu}(F) \text{ for every measurable cover } F \text{ of } E.$$

PROOF. (i) follows by an argument similar to the numerical anlogue in Halmos [4]. (ii) follows from (i) and (iii) follows from Theorem 5.4 and (ii) of the present proposition.

We state and prove the following main theorem of this section.

THEOREM 5.4. Let V be a weakly (σ, ∞)-distributive vector lattice and μ a σ-finite $V \cup \{\text{strict. } \infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set X. If μ^* is the outer measure induced by μ and if μ is outer regular, then $M_{\mu^*} = \widetilde{\mathcal{S}(\mathcal{R})}$ and μ^* on M_{μ^*} is the completion of $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$, where $\bar{\mu} = \mu^* | \mathcal{S}(\mathcal{R})$.

PROOF. Clearly $\widetilde{\mathcal{S}(\mathcal{R})} \subseteq M_{\mu^*}$ since μ^* is complete on M_{μ^*}. It is easy to check that $\mu^* | \widetilde{\mathcal{S}(\mathcal{R})} = \tilde{\mu}$, where $\tilde{\mu}$ is the completion of $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$. Thus it suffices to show that $M_{\mu^*} \subseteq \widetilde{\mathcal{S}(\mathcal{R})}$. Since by hypothesis μ is outer regular, and μ^* is σ-finite, the proof of this is similar to the numerical analogue in Halmos [4] and hence we omit details.

COROLLARY 5.6. If μ is a σ-finite $V \cup \{\text{strict. } \infty\}$-valued measure on a ring $\mathcal{R}$ of subsets of a set X and if V is a weakly (σ, ∞)-distributive vector lattice satisfying the countable chain condition then μ is outer regular, $M_{\mu^*} = \widetilde{\mathcal{S}(\mathcal{R})}$ and μ^* on M_{μ^*} is the completion of $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$, where $\bar{\mu} = \mu^* | \mathcal{S}(\mathcal{R})$.

PROOF. By Theorem 5.2 μ is outer regular. Now the corollary follows from the above theorem.

REMARK. The corresponding analogue of the above corollary for a σ-finite extended real valued measure μ on $\mathbb{R}$ is a consequence of the fact that $\mathbb{R}$ is a weakly (σ, ∞)-distributive vector lattice satisfying the countable chain condition and that μ is $\mathbb{R} \cup \{\text{strict. } \infty\}$-valued.

6. Some applications to positive operator valued measures in Banach spaces. The notion of positive operator valued measures (PO-measures in abbreviation) in Banach spaces has been introduced by us in [12]. In this section we give the Carathéodory extension of bounded PO-measures in Banach spaces as a particular case of Theorem 4.14 and consequently the Carathéodory extension of spectral measures in [10] is obtained as a corollary. Also we generalize here Theorem 6 of [10] to PO-measures when the Banach space is separable.

Before dealing with the applications, we give some definitions and results from [12] to make this section self-contained.

<u>DEFINITION 6.1</u> Let $\mathcal{R}$ be a ring of subsets of a set T. Let P(.) be a map: $\mathcal{R} \to W$, where W is a $W^*(\|.\|)$-algebra of operators on a Banach space X. (See [9] for definition of $W^*(\|.\|)$-algebras). Then P(.) is called a positive operator valued measure (abbrivated as PO-measure) on $\mathcal{R}$ if the range of P(.) is contained in $H(W)^+$ (the set of all real scalar type operators in W with spectrum contained in the set of non-negative reals), satisfying the following conditions:

(i) $P(\phi) = 0$;

(ii) $P(\bigcup_1^\infty \sigma_i)\, x = \lim_n \sum_{i=1}^n P(\sigma_i)\, x$, for each $x \in X$, where $\{\sigma_i\}$ is a disjoint sequence of sets in $\mathcal{R}$, with their union in $\mathcal{R}$.

Further, the PO-measure P(.) is said to be a spectral measure if the range of P(.) is contained in the set of all projections in W.

A PO-measure P(.) is said to be bounded if there exists a $T \in H(W)^+$ such that $P(\sigma) \leq T$ for all $\sigma \in \mathcal{R}$.

Throughout this section P(.) is a PO-measure with range in H(W), W a $W^*(\|.\|)$ algebra on X.

PROPOSITION 6.2. A PO-measure P(.) on $\mathcal{R}$ is a spectral measure if and only if P(.) is multiplicative.i.e $P(E \cap F) = P(E)\,P(F)$ for E, F $\varepsilon\ \mathcal{R}$.

PROPOSITION 6.3. H(W) is a boundedly complete vector lattice and P(.) is a PO-measure on $\mathcal{R}$ if and only if P(.) is a H(W)-valued measure in the sense of Definition 1.1. Further H(W) is hyperstonian and hence is a weakly (σ, ∞)-distributive Stone algebra.

PROPOSITION 6.4. Let $\mathcal{P}$ be a σ-complete B.A of projections on X (in the sense of Bade [1]). Let W be the algebra generated by $\mathcal{P}$ in the weark operator topology. Then W is a $W^*(\|.\|)$-algebra under a suitable equivalent norm $\|.\|$ on X. If X is weakly complete, it suffices to assume that $\mathcal{P}$ is a bounded B.A. of projections.

Now we study the applications of results in earlier sections to PO-measures in Banach spaces.

THEOREM 6.5. (Carathéodory extension theorem for bounded PO-measures) Let P(.) be a bounded PO-measure on a ring $\mathcal{R}$ of subsets of a set $\mathfrak{S}$ With its range contained in H(W).

(i) Then there is a unique bounded PO-measure $\overline{P}$ (.) on $\mathcal{S}(\mathcal{R})$ such that $\overline{P}(.)\,|_{\mathcal{R}} = P(.)$. Further, $\overline{P}(.)$ arises through the Carathéodory extension procedure (of §4.).

(ii) $\overline{P}(.)$ is spectral if and only if P(.) is spectral.

Consequently, every spectral measure $E(.)$ on $\mathcal{R}$ with its range contained in a σ-complete B.A. $\mathcal{P}$ of projections on X is extendable uniquely to a spectral measure $\overline{E}(.)$ on $\mathcal{S}(\mathcal{R})$, the σ-ring generated by $\mathcal{R}$, by the Carathéodory extension procedure and the range of $\overline{E}(.)$ is contained in $\overline{\mathcal{P}}^{s}$ (closure of $\mathcal{P}$ in the strong operator topology). If the Banach space X is weakly complete, it suffices to assume that P is bounded (in view of Proposition 6.4.).

<u>PROOF.</u> (i) The hypothesis of Theorem 4.14 are satisfied by $\mathcal{P}$ by Proposition 6.3 and hence by Theorem 4.14 there is a unique bounded H(W)-valued measure $\overline{P}(.)$ on $\mathcal{S}(\mathcal{R})$, extending $P(.)$. Further this extension arises by the Carathéodory extension procedure of §4. Again, as $\overline{P}(.)$ is a bounded H(W)-valued measure on $\mathcal{S}(\mathcal{R})$, $\overline{P}(.)$ is a bounded PO-measure on $\mathcal{S}(\mathcal{R})$ with range in H(W) by the first part of proposition 6.3. This proves (i).

(ii) It suffices to prove that $\overline{P}(.)$ is spectral if $P(.)$ is spectral. Let $P(.)$ be spectral. Then $P(\sigma)$ is a projection, for each $\sigma \in \mathcal{R}$. Let $\mathcal{B}_1 = \{P(\sigma) : \sigma \in \mathcal{R}\}$. Then $\mathcal{B}_1$ is a B.A. of projections and $\mathcal{B}_1 \subseteq \mathcal{B}$, the B.A of all projections in W. Since W is strongly closed, $\overline{\mathcal{B}}^{s} = \mathcal{B}$ and hence $\mathcal{B}$ is complete by Theorem 2.7 of Bade [1] and by the hypothesis that $\mathcal{B}$ is σ-complete. (See definition of $W^{*}(\|\cdot\|)$-algebras in [9]). From the definition of P^{*} and P_{*} (corresponding to μ^{*} and μ_{*} respectively in §4) it is clear that the ranges of $P_{*}(.)$ and $P^{*}(.)$ are contained in $\mathcal{B}$, as $\mathcal{B}$ is complete. Thus P^{*} and hence $\overline{P}$ are projection valued. i.e. $\overline{P}(.)$ is a spectral measure on $\mathcal{S}(\mathcal{R})$.

For proving the last part of the theorem, let W be the weakly closed algebra generated by $\bar{\mathcal{P}}^s$, which is a complete B.A. of projections by Theorem 2.7 of Bade. [1]. By Proposition 6.4 W is a $W^*(\|\,.\,\|)$-algebra under a suitable equivalent norm $\|\,.\,\|$ on X and E(.) is a spectral PO-measure on $\mathcal{R}$ with its range contained in $\bar{\mathcal{P}}^s \subsetneq H(W)$. Now from (i) and (ii) of the theorem and from the fact that $\bar{\mathcal{P}}^s$ is the collection of all projections in W, the last part of the theorem follows.

REMARK. The above theorem is clearly a generalization of Theorem 7 of Berbarian [2] to Banach spaces when the operators in the range of the PO-measure then commute with each other.

THEOREM 6.6. If P(.) is a bounded PO-measure in a separable Banach space X defined on a ring $\mathcal{R}$ of subsets of a set T, with its range contained in H(W), then P(.) is outer regular in the sense of Definition 5.1. Further, $M_{\rho^*} = \tilde{\mathcal{S}}(\mathcal{R})$ and $P^*(.)$ on M_{ρ^*} is the completion of $\bar{P}$ on $\mathcal{S}(\mathcal{R})$, where $\bar{P}(.) = P^*(.) \mid \mathcal{S}(\mathcal{R})$. ($P^*(.)$ is the outer measure induced by P(.)).

PROOF. By Proposition 6.3, P(.) is a H(W)-valued bounded measure (in the sense of Definition 1.1) and H(W) is a weakly (σ, ∞)-distributive Stone algebra. Since the latter part of the theorem follows from the outer regularity of P(.) in view of Theorem 5.4, it suffices to prove that P(.) is outer regular.

Since P(.) is a bounded PO-measure, there exists $T \varepsilon H(W)^+$ such that $P(\sigma) \leq T$ for every $\sigma \varepsilon \mathcal{R}$. From the definition of partial ordering in H(W) it is clear that $\| P(\sigma)\| \leq \| T \|$, where $\| S\| = \sup_{\| x\| = 1} \| Sx\|$, $\| . \|$ on X being that one occuring in the definition of the $W^*(\| . \|)$-algebra W.

The outer regularity of P(.) can be proved exactly on the same lines of the proof of Theorem 5.2, if we can show that for each decreasing net $\{T_\alpha\}$ of operators in $H(W)^+$ which is norm bounded, there exists a sequence $\{T_n\}$ such that $\{T_n\} \subseteq \{T_\alpha\}$ and $\bigwedge_n T_n = \bigwedge_\alpha T_\alpha$ in H(W). For this, because of Theorem 3 of [9], it suffices to show that a sequence $\{T_n\} \subseteq \{T_\alpha\}$ exists such that $\lim_n T_n x = \lim_\alpha T_\alpha x$, $x \varepsilon X$. But, since X is separable, by following an argument similar to the classical Hilbert space case it can be shown that on norm bounded sets of H(W) the strong operator topology is metrisable. Consequently, as the range of P(.) is norm bounded in H(W), the result follows

<u>REMARK</u>. The above theorem generalizes Theorem 6 of [10] to PO-measures in separable Banach spaces. We also remark that the proof of Theorem 6 in [10] is erroneous as Theorem 5 of Lumer [7] does not apply there. We do not know whether Theorem 6 of [7] is still valid without the additional hypothesis of separability of the Banach space.

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