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A GENERALISED SPECTRAL MAPPING THEOREM

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ABSTRACT

The classical notion of positive operator valued measures in Hilbert spaces is extended here to Banach spaces and a generalized spectral mapping theorem is obtained for spectral measures in Banach spaces.

Introduction. A spectral measure in Hilbert space can be considered as a positive operator valued measure whose range is contained in an abelian von Neumann algebra of operators. In [10], one of the authors extended the notions of abelian von Neumann algebras to Banach spaces, by introducing the new concept of $W^*(\|\cdot\|)$-algebras. The present paper generalises the notion of positive operator valued measures to Banach spaces in the set up of $W^*(\|\cdot\|)$-algebras.

The first section deals with some basic definitions and results from literature to facilitate an easy understanding of the paper. In §2 the notion of positive operator valued measures in Banach spaces is introduced and a necessary and sufficient condition is given for a positive operator valued measures to be a spectral measure. In §3 it is shown that our operator valued measure gives a concrete example of a V-valued measure (in the sense of Wright [13]) with V a weakly (σ, ∞)-distributive boundedly complete vector lattice. The last section deals with some generalizations of results in §7 of [2] to positive operator valued measures in Banach spaces, with the main result being the one giving a generalization of the classical spectral mapping theorem.

1. Preliminaries. Throughout this paper X will denote a complex Banach space. By an operator on X we mean a bounded linear mapping of X into itself and B(X) will denote the collection of all such operators on X.

If $|||\cdot|||$ is the norm of X, one can define an operator T on X to be hermitian in $|||\cdot|||$ if $||| I + irT ||| = 1 + o(r)$ as $r \to 0$, where I is the identity operator on X. A detailed study of such hermitian operators can be found in Vidav [15] and Berkson [3,4]. Lumer introduced in [9] the notion of a semi-inner-product [,] induced by the norm $|||\cdot|||$ of X and proved in [9] that T is hermitian in $|||\cdot|||$ if and only if $[Tx, x]$ is real for $|||x||| = 1$, x in X, where [,] is a semi-inner-product induced by $|||\cdot|||$.

An idempotent operator P on X is called a projection. We refer to Bade [1] for the definitions of bounded, σ-complete and complete Boolean algebras (B.A.) of projections on X.

From Panchapagesan [10] we give below the definition of a $W^*(\|\cdot\|)$-algebra, which plays a key role in the present work.

DEFINITION 1. By a $W^*(\|\cdot\|)$-algebra W on a Banach space X we mean a pair, consisting of a commutative weakly closed subalgebra of W of B(X) and an equivalent norm $\|\cdot\|$ on X, such that the set of all projections in W is a σ-complete B.A. of projections on X and such that every element S in W has the representation of the form $S = R + iJ$ where R and J satisfy the following conditions:

(i) $RJ = JR$;

(ii) $R^m J^n$ ($m,n = 0,1,2,\ldots$) are hermitian in the norm $\|\cdot\|$;

(iii) R and J are in W.

One may refer to [10] for the basic properties of such $W^*(\|\cdot\|)$-algebras.

Again we recall from [10] the following definition of positive operators on X.

DEFINITION 2. An operator T on X is said to be positive in the equivalent norm $\|\cdot\|$ on X (and we denote this by $T \geq 0$ in $\|\cdot\|$) if $[Tx, x] \geq 0$ for x in X with $\|x\| = 1$, where $[\ ,\ ]$ is a semi-inner-product induced by the norm $\|\cdot\|$ on X. Again for two operators T_1 and T_2 on X, we say $T_1 \geq T_2$ in the equivalent norm $\|\cdot\|$ on X (briefly, $T_1 \geq T_2$ in $\|\cdot\|$) if (i) T_1, T_2 are hermitian in $\|\cdot\|$ and (ii) $T_1 - T_2 \geq 0$ in $\|\cdot\|$.

Throughout this paper W will denote a $W^*(\|\cdot\|)$-algebra of operators on X and $H(W)$ will be the set of all operators in W, which are hermitian in $\|\cdot\|$. Let $H(W)^+$ denote the set of all operators A in $H(W)$ such that $A \geq 0$ in $\|\cdot\|$. Then $H(W)$ is a boundedly complete vector lattice under the partial ordering of Definition 2. (Theorem 3 of [10]).

2. Positive operator valued measures in Banach spaces. In this section we generalize the classical notion of positive operator valued measures with commuting range to Banach spaces.

DEFINITION 3. Let $\mathcal{R}$ be a ring of subsets of a set $\mathfrak{S}$. Let $P(.)$ be a map: $\mathcal{R} \to W$, where W is a $W^*(\|\cdot\|)$-algebra of operators on the Banach space X. Then $P(.)$ is called a positive operator valued measure (abbreviated as PO-measure) on $\mathcal{R}$ if the range of $P(.)$ is contained in $H(W)^+$, satisfying the following conditions:

(i) $P(\phi) = 0$;

(ii) $P(\bigcup_{i=1}^{\infty}\sigma_i)\, x = \lim_n \sum_{i=1}^{n} P(\sigma_i)x$,

for each x in X, where $\{\sigma_i\}_1^{\infty}$ is a sequence of pairwise disjoint sets in $\mathcal{R}$, with their union in $\mathcal{R}$.

Further, the PO-measure $P(.)$ is said to be a spectral measure on $\mathcal{R}$ if the range of $P(.)$ is contained in the set of all projections in W.

Note that a PO-measure $P(.)$ is monotonic and subtractive.

PROPOSITION 1. A PO-measure $P(.)$ is a spectral measure on $\mathcal{R}$ if and only if $P(.)$ is multiplicative.

PROOF. If $P(.)$ is multiplicative, then $P(\sigma) = P^2(\sigma)$ so that $P(\sigma)$ is a projection in W for each σ in $\mathcal{R}$.

Conversely, if $P(.)$ is a spectral measure, then for σ, δ in $\mathcal{R}$, by the additivity of $P(.)$ we have

$$P(\sigma \cup \delta) = P(\sigma) + P(\delta \setminus \sigma) = P(\delta) + P(\sigma \setminus \delta).$$

By monotoneity of $P(.)$

$$P(\sigma) = P(\sigma)P(\sigma \cup \delta) = P(\sigma)P(\delta) + P(\sigma \setminus \delta)$$

and

$$P(\delta) = P(\delta)P(\sigma \cup \delta) = P(\delta)P(\sigma) + P(\delta \setminus \sigma).$$

Thus

$$P(\sigma) - P(\sigma \setminus \delta) + P(\delta) - P(\delta \setminus \sigma) = 2\, P(\sigma)P(\delta).$$

By the subtractivity of $P(.)$ we have

$$P(\sigma \cap \delta) = P(\sigma)P(\delta).$$

Thus $P(.)$ is multiplicative.

3. PO-measures as vector lattice valued measures. The present section is devoted to show that a PO-measure is also an $H(W)$-valued measure in the sense of Wright (Definition on p.193 of [13]) and to prove that $H(W)$ is a weakly (σ, ∞)-distributive boundedly complete vector lattice. See p.279 of [14] for the definition of weakly (σ, ∞)-distributive vector lattices.

We recall the following definition from Wright [13].

DEFINITION 4. A V-valued measure on the ring $\mathcal{R}$ of subsets of a set $\mathcal{E}$ is a map $\mu : \mathcal{R} \to V$, where V is a boundedly complete vector

... such that

(i) $\mu(\sigma) \geq 0$ for every σ in $\mathcal{R}$;

(ii) $\mu(\varphi) = 0$;

(iii) $\mu(\bigcup_1^\infty \sigma_i) = \bigvee_{n=1}^\infty \sum_{i=1}^n \mu(\sigma_i)$, where $\{\sigma_i\}_1^\infty$ is a sequence of pairwise disjoint sets in $\mathcal{R}$ with $\bigcup_1^\infty \sigma_i$ in $\mathcal{R}$.

LEMMA 1. Let P(.) be a PO-measure on $\mathcal{R}$, a ring of subsets of a set $\mathfrak{S}$, with its range in H(W). Then P(.) is a H(W)-valued measure on $\mathcal{R}$ in the sense of Definition 4. Conversely, any H(W)-valued measure on $\mathcal{R}$ in the sense of Definition 4 is a PO-measure.

PROOF. By Theorem 3 of Panchapagesan[10], H(W) is a boundedly complete vector lattice with the strong operator topology on H(W) compatible with its order structure. Let P(.) be a PO-measure on $\mathcal{R}$ with its range in H(W). Let $\{\sigma_i\}_1^\infty$ be a pairwise disjoint sequence of sets in $\mathcal{R}$ with $\bigcup_{i=1}^\infty \sigma_i \in \mathcal{R}$. By monotoneity of P(.)

$$P(\bigcup_1^\infty \sigma_i) \geq P(\bigcup_1^n \sigma_i) = \sum_1^n P(\sigma_i) \quad \text{in } \|\cdot\|$$

for each n. Consequently, $\{\sum_{i=1}^n P(\sigma_i)\}$ is order bounded above and hence by Theorem 3 of [10]

$$\lim_n \left[\sum_{i=1}^n P(\sigma_i)x\right] = \left(\bigvee_{n=1}^\infty \sum_{i=1}^n P(\sigma_i)\right) x$$

for each x in X. Thus

$$P(\bigcup_{i=1}^\infty \sigma_i) = \bigvee_{n=1}^\infty \sum_{i=1}^n P(\sigma_i) \quad .$$

Since the above arguments are also reversible in light of Theorem 3 of [10], the converse part of the lemma follows.

In light of Theorem 18 of Stone [12] one may define a compact Hausdorff space S to be stonian if S is extremally disconnected. Following Dixmier [5], we call a stonian space S hyperstonian if the union of the supports of all positive normal Borel measures on S is everywhere dense in S.

Next we prove the following interesting result for $W^*(\|\cdot\|)$- algebras which generalizes the corresponding one for abelian von Neumann algebras found in [5].

LEMMA 2. The maximal ideal space S of a $W^*(\|\cdot\|)$-algebra W is hyperstonian.

PROOF. From Definition 1 the set $\mathcal{P}$ of all projections in W is a σ-complete B.A. of projections and hence its strong closure $\bar{\mathcal{P}}^s$ is a complete B.A. of projections by Theorem 2.7 of Bade [1] . Since W is also strongly closed, $\bar{\mathcal{P}}^s = \mathcal{P}$ so that $\mathcal{P}$ is complete. Thus W is weakly generated by the complete B.A. $\mathcal{P}$ of projections and hence by another theorem of Bade [1] W is the uniformly closed algebra $\mathcal{A}(\mathcal{P})$ generated by $\mathcal{P}$. Hence by a well-known result on commutative Banach algebras the maximal ideal space S of W is homeomorphic to the Stone's representation space of the complete B.A. $\mathcal{P}$ so that S is extremally disconnected by Theorem 18 of Stone [12].

To show that S is hyperstonian we adapt Dixmier's argument in [5]. Let $[\ ,\ ]$ be a semi-inner-product on X induced by the norm $\|\cdot\|$ (given in the $W^*(\|\cdot\|)$-algebra W). By Theorem 3 of Panchapagesan [10] for every increasing net $\{T_\alpha\}$ of operators in H(W), order bounded above,

$$(\bigvee_\alpha T_\alpha)x = \lim_\alpha T_\alpha x$$

for each x in X and hence

$$[\bigvee_\alpha T_\alpha x, x] = \lim_\alpha [T_\alpha x, x] = \sup_\alpha [T_\alpha x, x]$$

since $[\cdot x, x]$ is a bounded linear functional on B(X). Consequently, each x in X gives rise to a normal continuous positive linear functional $[\cdot x, x]$ on H(W) and hence on the isometric isomorphic image C(S), the Banach algebra of all real valued continuous functions on S. Hence by the Riesz representation theorem this corresponds to a unique regular a positive measure μ_x on C(S) which is further normal. The union of the supports of all the measures μ_x, x in X, is dense in S. For, otherwise, let G be a non-null clopen subset of S disjoint with this union. Let χ_G be the characteristic function of G. Then χ_G is in C(S) and if χ_G corresponds to the projection E in H(W), then $E \neq 0$ and hence by Theorem 5 of [9], there exists $y \neq 0$ in X such that $[Ey, y] \neq 0$. This is a contradiction since the normal measure μ_y does not vanish on G. Hence S is hyperstonian.

COROLLARY. Let $\mathcal{P}$ be a σ-complete B.A. of projections on X. Let W be the algebra generated by $\mathcal{P}$ in the weak operator topology. If S is the maximal ideal space of W, then S is hyperstonian. If X is weakly complete, it suffices to assume that $\mathcal{P}$ is a bounded B.A. .

PROOF. The corollary follows from (1.2) of Panchapagesan [11] and the above lemma.

In [14] Wright studied the properties of certain classes of V-valued measures when V is a weakly (σ , ∞)-distributive vector lattice. From his results it follows that in Hilbert space the classical PO-measures, with the operators in the range commuting with each other, are concrete examples of V-valued measures with V weakly (σ, ∞)-distributive. The above lemmas give rise to another concrete example of such V-valued measures, viz. the PO-measures with range in H(W). This fact is stated as the following theorem.

THEOREM 1. Let P(.) be a PO-measure on $\mathcal{R}$, a ring of subsets of a set $\mathfrak{S}$, with its range contained in H(W). Then P(.) is a H(W)-valued measure on $\mathcal{R}$ in the sense of Wright (Definition 4) and H(W) is a weakly (σ, ∞)-distributive boundedly complete vector lattice.

PROOF. The theorem follows from Lemmas 1 and 2 and from Lemma 2.3 of Wright [14] and from the fact that in a hyperstonean space every meagre set is nowhere dense. (See Corollary to Proposition 5 of Dixmier [5]).

4. A generalised spectral mapping theorem. In this section we obtain some generalisations of the results in § 7 of Berberian [2] to PO-measures in Banach spaces. The main result is a generalisation of the classical spectral mapping theorem.

Throughout this section T will denote a locally compact Hausdorff space and $\mathcal{B}_W$ will be the σ-algebra generated by the class of all open sets in T. $\mathcal{C}$ will denote the collection of all compact sets in T. A PO-measure on $\mathcal{B}_W$ is called a weakly Borel PO-measure on T.

DEFINITION 5. A weakly Borel PO-measure P(.) on T with its range in H(W) is said to be regular if for every A in $\mathcal{B}_W$ and for every x in X

$$P(A)x = \lim \left\{ P(C)x : A \supseteq C \in \mathcal{C} \right\}.$$

DEFINITION 6. If P(.) is a weakly Borel PO-measure on T, the co-spectrum of P(.) is defined to be the union of all open sets in T on which P(.) vanishes and the complement of the co-spectrum is called the spectrum or support of P(.). P(.) is said to be compact if the spectrum of P(.) is a compact set in T. P(.) is said to be normalised if P(T) = I. We denote by $\Lambda(P)$ the spectrum of P(.).

DEFINITION 7. Let P(.) be a H(W)-valued weakly Borel spectral measure on T and f be a bounded measurable function on the spectrum of P(.). Then the spectral norm of f, denoted by $N_P(f)$, is defined as

$$N_P(f) = \sup \left\{ |f(s)| : s \in \Lambda(P) \right\} .$$

LEMMA 3. Let P(.) be a regular weakly Borel normalised compact spectral measure on T, with its range in H(W) on X. Then

$$\left\| \int_T f \, dP \right\| = N_P(f)$$

for every continuous complex valued function on $\Lambda(P)$ where

$$\| A \| = \sup\{ \| Ax \| : x \text{ in } X, \| x \| = 1 \}$$

and $\| \cdot \|$ is the norm in the $W^*(\| \cdot \|)$- algebra W.

PROOF. Refer to p.2105 of Dunford and Schwartz [7].

LEMMA 4. If $P(.)$ is a regular weakly Borel ~~xxxxxxx~~ PO-measure on T with its range in H(W) on X, then $P(T) = P($ spectrum of $\Lambda(P)$).

PROOF. The proof is quite similar to that of Theorem 23 of Berberian [2] and hence omitted.

LEMMA 5. If $P(.)$ is a regular weakly Borel compact normalised spectral measure on T with its range cotained in H(W) on X and if g is a continuous complex function on T, which vanishes at some point of $\Lambda(P)$, then the operator $\int_T g \, dP$ is singular.

PROOF. The proof of Lemma on p.58 of Berberian [2] holds here verbatim, by making the following alterations. The vector x on p.58 of [2] should be chosen with $P(M)x = x$ and $\| x \| = 1$, where $M = \{ \lambda \in T : |g(\lambda)|^2 < \varepsilon/2 \}$ and $\| \cdot \|$ is the norm given in the $W^*(\| \cdot \|)$-algebra W. The inner product on p.59 of [2] should be replaced by a semi-inner-product induced by the norm $\| \cdot \|$ in W.

THEOREM 2. (Generalised spectral mapping theorem). If $P(.)$ is a normalised regular weakly Borel compact spectral measure on the locally compact Hausdorff space T, with its range cotained in a $W^*(\| . \|)$-algebra W on the Banach space X, then

$$\sigma\left(\int_T f \, dP \right) = f(\Lambda(P)),$$

for every complex valued continuous function f on T, where $\sigma(A)$ denotes the spectrum of the operator A.

PROOF. As $P(.)$ is a normalised spectral measure, by Lemma 6 of Dunford [6] $\int_T f \, dP$ is a scalar type operator. Then by Theorem 5

of Foguel [8], $\pi(\int_T f\,dP) = \sigma(\int_T f\,dP)$
where $\pi(A)$ is the approximate point spectrum of A. This observation together with the arguments on p.60 of Berberian [2] will show that $f(\Lambda(P)) \subseteq \sigma(\int_T f\,dP)$.

The reverse inclusion relation can be proved by an argument quite similar to that on p.61 of [2] and by the fact that W is a B*-algebra in the operator norm induced by the norm $\|\cdot\|$ in its definition. This completes the proof of the theorem.

COROLLARY 1. If $P(.)$ is a countably additive regular spectral measure (in the sense of Dunford and Schwartz [7]) with compact support $\Lambda(P)$ on the locally compact Hausdorff space T, then

$$\sigma(\int_T f\,dP) = f(\Lambda(P))$$

for every complex valued continuous function f on T, where $\sigma(A)$ denotes the spectrum of the operator A.

PROOF. By Corollary XVII.3.10 of Dunford and Schwartz [7] the range of $P(.)$ is a σ-complete B.A. of projections. Let us denote this range by $\mathcal{P}$. By (1.2) of Panchapagesan [11], the weakly closed algebra generated by $\mathcal{P}$ is a $W^*(\|\cdot\|)$-algebra for some suitable equivalent norm $\|\cdot\|$ on X and hence the corollary follows from the above theorem.

COROLLARY 2. If $P(.)$ is a normalised regular weakly Borel compact spectral measure on the locally compact Hausdorff space T with its range contained in a $W^*(\|\cdot\|)$-algebra W on X, then

$$\|\int_T f\,dP\| = \sup\left\{|\lambda| : \lambda \text{ in } \sigma(\int_T f\,dP)\right\}$$

for every complex valued continuous function f on T.

PROOF. The corollary follows from the above theorem and Lemma 3.

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