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"A CHARACTERIZATION OF THE ENTROPY FUNTIONALS FOR GRAND CANONICAL ENSAMBLES THE DISCRETE CASE"

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A CHARACTERIZATION OF THE ENTROPY FUNCTIONALS FOR GRAND CANONICAL ENSEMBLES. THE DISCRETE CASE.

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SUMMARY.- "Natural" properties lead to a representation theorem - for the entropy functional of a grand canonical ensemble. In addition to the classical Boltzmann term, the representation contains three more terms that seem to be meaningful in statistical mechanics.

SUNTO.- Da proprieta "naturali" imposte al funzionale entropia di un insieme gran canonico si deduce un teorema di rappresentazione - per esso. In aggiunta al termine classico di Boltzmann, la rappresentazione mette in luce tre ulteriori termini che sembrano significativi in meccanica statistica.

(*) *While on sabbatical leave of absence from the University of - Waterloo, Canada, Department of Applied Mathematics.*

1.- Introduction.

The purpose of this paper is to derive the representation of the entropy functionals for grand canonical ensembles in statistical mechanics.

The (mean) entropy for classical systems, in the grand canonical description, has been defined essentially by (see ref. 1):

$$(1.1)\qquad S(\pi) = -k \sum_{0}^{\infty}{}_{n}\, p_n \sum_{0}^{m_n}{}_{i}\, q_{ni} \log(p_n q_{ni}) \qquad ,$$

in the discrete case, and by

$$(1.2)\qquad S(f) = -k \sum_{1}^{\infty}{}_{n}\, p_n \int_{R_n} f_n \log f_n \, dx_1 \ldots dx_n - k \sum_{0}^{\infty} p_n \log p_n ,$$

in the continuous case, where $q_{ni} > 0$ for every $n \geq 0$, $0 \leq i \leq m_n$, $p_j \geq 0$ for every $j \geq 0$, $p_n = 0$ for n sufficiently large, and

$$\sum_{0}^{\infty}{}_{n}\, p_n = 1 , \qquad \sum_{0}^{m_n}{}_{i}\, q_{ni} = 1 \quad \text{for all} \quad n \geq 0 ;$$

for every $n > 0$, f_n is a (conditional) probability density on R_n s.t. $f_n \log f_n$ is integrable, k is a nonnegative real number.

In ref. 1 properties of the mean entropy have been investigated. But so far no attempt has been done to derive from those properties the representations (1.1) and/or (1.2). Moreover, the properties which have been considered in ref. 1 seem to be more "natural" in communication theory rather than in statistical mechanics (see ref. 2).

By imposing certain conditions, that seem to be intuitively "natural" in statistical mechanics, on the unknown entropy we shall derive the following forms

$$(1.3) \qquad S(\pi) = -a \sum_{0}^{\infty}{}_{n}\, p_n \sum_{0}^{m_n}{}_{i}\, q_{ni} \log q_{ni} + b \sum_{0}^{\infty}{}_{n}\, p_n\, n + c \sum_{0}^{\infty}{}_{n}\, p_n \log m_n$$

$$- d \sum_{0}^{\infty}{}_{n}\, p_n \log p_n$$

in the discrete case, and

$$(1.4) \qquad S(f) = - a \sum_{1}^{\infty}{}_{n}\, p_n \int_{R_n} f_n \log f_n \, dx_1 \ldots dx_n + b \sum_{0}^{\infty}{}_{n}\, p_n\, n +$$

$$c \sum_{1}^{\infty}{}_{n}\, p_n \log m\,(A_{f_n}) - d \sum_{0}^{\infty}{}_{n}\, p_n \log p_n \qquad ,$$

in the continuous case, where $A_{f_n} := \{ x \in R_n \mid f_n(x) > 0 \}$, and a, b, c, d are real numbers, with $a \geq 0$, $c \geq 0$, $d \geq 0$.

These representations are consistent with the representations that have been derived in ref. 3 and ref. 4 for the canonical ensembles.

Note that the last three terms on the rhs. of (1.3) and (1.4) are missing in the representations (1.1), (1.2) and we believe that they have some significance. This paper will deal with the discrete case.

2.- Preliminaries.

Let N denote the set of all nonnegative integers. Let $\pi(1) := \{ \pi_{ij} = p_i\, q_{ij},\ i \in N,\ j \in N \}$ with $p_i \geq 0$ for all $i \in N$, $q_{ij} \geq 0$ for all $i, j \in N$,

$$(2.1) \qquad \sum_{0}^{\infty}{}_{i}\, p_i = 1 \quad , \qquad \sum_{0}^{\infty}{}_{j}\, q_{ij} = 1 \qquad \text{for all} \quad i \in N,$$

be any discrete probability distribution s.t. $p_i = 0$ for i sufficiently large, $q_{00} = 1$, $q_{0j} = 0$ for all $j > 0$, $q_{ij} = 0$ for every $i > 0$ and j sufficiently large. $W(1)$ will denote the set of all such probability distributions.

We shall denote by $\pi(2) := \{ \pi_{risj} = p_{rs}q_{rsij}, r \in N, i \in N, s \in N, j \in N \}$ with $p_{rs} \geq 0$, $q_{rsij} \geq 0$ for all $r, s, i, j \in N$ and

$$(2.2) \quad \sum_{0}^{\infty}{}_{r} \sum_{0}^{\infty}{}_{s} p_{rs} = 1, \quad \sum_{0}^{\infty}{}_{i} \sum_{0}^{\infty}{}_{j} q_{rsij} = 1 \quad \text{for all } r, s \in N,$$

any discrete probability distribution s.t. $p_{rs} = 0$ for r and s sufficiently large, $q_{oooo} = 1$, $q_{osij} = 0$ for all $i > 0$, $s \in N$, $j \in N$, $q_{roij} = 0$ for all $j > 0$, $r \in N$, i N, $q_{rsij} = 0$ for every $r > 0$, $s > 0$ and i, j suffcicieently large. Again $W(2)$ will denote the set of all such probability distributions.

If a probability distribution $\pi(2) \in W(2)$ is such that $p_{rs} = 0$ for all $s > 0$ and each $r \in N$ (or $p_{rs} = 0$ for all $r > 0$ and each $s \in N$) we shall identify $\pi(2)$ to the marginal distribution $\pi'(1)$ (to the marginal distribution $\pi''(1)$, respectively) hence to the distribution $p'_r := p_{ro}$ and $q'_{ij} := q_{iojo}$ for each $r, i, j \in N$ ($p''_s := p_{os}$, $q''_{ij} := q_{oioj}$ for each $s, i, j \in N$).

Consider the mapping S of $W(2)$ into R defined by

$$(2.3) \quad S(\pi(2)) = -a \sum_{0}^{\infty}{}_{r} \sum_{0}^{\infty}{}_{s} p_{rs} \sum_{0}^{\infty}{}_{i} \sum_{0}^{\infty}{}_{j} q_{rsij} \log q_{rsij}$$

$$+ \sum_{0}^{\infty}{}_{r} \sum_{0}^{\infty}{}_{s} p_{rs}(b_r + b_s) + c \sum_{0}^{\infty}{}_{r} \sum_{0}^{\infty}{}_{s} p_{rs} \log m_{rs}$$

$$-d \sum_0^\infty {}_r \sum_0^\infty {}_s \, p_{rs} \log p_{rs} \qquad ,$$

where a, b_r $(r = 0, 1, \ldots)$, c, d are real numbers, $a \geq 0$, $b_0 = 0$, $c \geq 0$, $d \geq 0$ and with the convention $0 \log 0 := 0$; for every fixed couple (r,s) in $N \times N$, m_{rs} is the total number of $\{ q_{rsij}, i \in N, j \in N \}$ that are greater than 0.

Clearly, when $\pi(2)$ reduces to $\pi(1)$ and $b_n = bn$, (2.3) reduces to (1.3).

The (entropy) function S has the following properties:

(1) Let $\pi(2) \in W(2)$ and let $\pi'(1)$, $\pi''(1)$ be its marginal distributions, defined by

(2.4) $$\pi'(1) := \{ \pi'_{ri} = \sum_0^\infty {}_s \sum_0^\infty {}_j \, p_{rs} q_{rsij}, \; r \in N, \; i \in N \}$$

$$\pi''(1) := \{ \pi''_{sj} = \sum_0^\infty {}_r \sum_0^\infty {}_i \, p_{rs} q_{rsij}, \; s \in N, \; j \in N \} \quad ,$$

then

$$S(\pi(2)) \leq S_1(\pi'(1)) + S_1(\pi''(1)) \qquad \text{(subadditivity)} \; ,$$

where S_1 denotes the restriction of S to $W(1)$.

(2) If for every $r, s, i, j \in N$: $\pi_{risj} = \pi'_{ri} \; \pi''_{sj}$, with π'_{ri} and π''_{sj} defined as in Eq. (2.4), then

$$S(\pi(2)) = S_1(\pi'(1)) + S_1(\pi''(1)) \qquad \text{(additivity)}.$$

(3a) For every fixed (r,s) in $N \times N$: $\{ q_{rsij}, i \in N, j \in N\} \to S(\pi(2))$ is symmetric.

(3b) On the set of all probability distributions in $W(2)$ s.t. $q_{rsij} = q_{srji}$, for all $i, j \in N$, $(r,s) \to S(\pi(2))$ is symmetric.

(4) Let $\pi_{x,m,n}(1) := \{\pi_{ri} = p_r q_{ri},\ r \in N,\ i \in N\}$, with $x \in [0,1]$, $0 \leq m < n$, be such that $p_r = 0$ for all $r \neq m,n$, $p_m = 1-x$, $p_n = x$; the function

$$\psi_{m,n}(x) := S_1(\pi_{x,m,n}(1)) \qquad (x \in [0,1])$$

is continuous at 0.

<u>Proof</u>: Properties (2), (3a), (3b) and (4) follow immediately from Eq. (2.3).

But it is worthwhile to give a detailed although easy proof of property (1). We shall prove indeed that each term on the rhs of (2.3) is subadditive. We start with the first term

$$- a \sum_0^\infty {}_r \sum_0^\infty {}_s\, p_{rs} \sum_0^\infty {}_i \sum_0^\infty {}_j\, q_{rsij} \log q_{rsij} \qquad ;$$

we introduce the following definitions

$$p'_r := \sum_0^\infty {}_s\, p_{rs} \quad , \qquad p''_s := \sum_0^\infty {}_r\, p_{rs} \qquad ;$$

if for some r or some s in N : $p'_r = 0$ or $p''_s = 0$, then $\{ q'_{ri},\ i \in N \}$ and $\{ q''_{sj},\ j \in N \}$ are any probability distributions, otherwise

$$q'_{ri} := \frac{\sum_0^\infty {}_s\, p_{rs} \sum_0^\infty {}_j\, q_{rsij}}{p'_r}$$

$$q''_{sj} := \frac{\sum_0^\infty {}_r\, p_{rs} \sum_0^\infty {}_i\, q_{rsij}}{p''_s}$$

We have to prove that the following inequality holds

$$(2.5)\quad -a\sum_{0}^{\infty}{}_{r,s}\, p_{rs} \sum_{0}^{\infty}{}_{i,j}\, q_{rsij}\log q_{rsij} + a\sum_{0}^{\infty}{}_{r}\, p'_r \sum_{0}^{\infty}{}_{i}\, q'_{ri}\log q'_{ri}$$

$$+a\sum_{0}^{\infty}{}_{s}\, p''_s \sum_{0}^{\infty}{}_{j}\, q''_{sj} \log q''_{sj} \leq 0 \quad ,$$

with $a \geq 0$.

The convexity of the function $x \log x$ implies

$$(2.6)\quad p'_r\, q'_{ri} \log q'_{ri} \leq \sum_{0}^{\infty}{}_{s}\, p_{rs} \left(\sum_{0}^{\infty}{}_{j}\, q_{rsij}\right) \log \left(\sum_{0}^{\infty}{}_{k}\, q_{rsik}\right)$$

$$p''_s\, q''_{sj} \log q''_{sj} \leq \sum_{0}^{\infty}{}_{r}\, p_{rs} \left(\sum_{0}^{\infty}{}_{i}\, q_{rsij}\right) \log \left(\sum_{0}^{\infty}{}_{h}\, q_{rshj}\right) \quad ,$$

for each r and s in N such that $p'_r \neq 0$ and $p''_s \neq 0$ and - every $i, j \in N$.

Since $p'_r = 0$ and $p''_s = 0$ imply $p_{rs} = 0$ for all $s \in N$ and $p_{rs} = 0$ for all $r \in N$, respectively, if for some r in N or some s in N : $p'_r = 0$ or $p''_s = 0$, inequalities (2.6) still hold. Hence

$$(2.7)\quad -a\sum_{0}^{\infty}{}_{r}\sum_{0}^{\infty}{}_{s}\, p_{rs} \sum_{0}^{\infty}{}_{i}\sum_{0}^{\infty}{}_{j}\, q_{rsij}\log q_{rsij} + a\sum_{0}^{\infty}{}_{r}\, p'_r \sum_{0} q'_{ri}\log q'_{ri}$$

$$+a\sum_{0}^{\infty}{}_{s}\, p''_s \sum_{0}^{\infty}{}_{j}\, q''_{sj} \log q''_{sj} \leq$$

$$a \sum_{0}^{\infty}{}_{r} \sum_{0}^{\infty}{}_{s}\, p_{rs} \sum_{0}^{\infty *}{}_{i,j}\, q_{rsij} \log \frac{\left(\sum_{0}^{\infty}{}_{k}\, q_{rsik}\right)\left(\sum_{0}^{\infty}{}_{h}\, q_{rshj}\right)}{q_{rsij}}$$

where $\sum_{0}^{\infty *}{}_{i,j}$ denotes the sum over the set of indices $\{ i,j \in N \mid q_{rsij} \neq 0\}$. By using the inequality

$$\log x \leq x - 1 \quad \forall \quad x > 0,$$

we have

$$(2.8) \quad a \sum_{0}^{\infty}{}_{r} \sum_{0}^{\infty}{}_{s} p_{rs} \sum_{0}^{\infty *}{}_{i,j} q_{rsij} \log \frac{(\sum_{0}^{\infty}{}_{k} q_{rsik})(\sum_{0}^{\infty}{}_{h} q_{rshj})}{q_{rsij}} \leq$$

$$a \sum_{0}^{\infty}{}_{r} \sum_{0}^{\infty}{}_{s} p_{rs} \sum_{0}^{\infty}{}_{i} \sum_{0}^{\infty}{}_{j}((\sum_{0}^{\infty}{}_{k} q_{rsik})(\sum_{0}^{\infty}{}_{h} q_{rshj}) - q_{rsij})=0.$$

By combining (2.7) and (2.8) we obtain (2.5).

The second term on the rhs of (2.3) is additive and therefore - subadditive too. Indeed

$$\sum_{0}^{\infty}{}_{r} \sum_{0}^{\infty}{}_{s} p_{rs}(b_r+b_s) = \sum_{0}^{\infty}{}_{r}(\sum_{0}^{\infty}{}_{s} p_{rs})b_r + \sum_{0}^{\infty}{}_{s}(\sum_{0}^{\infty}{}_{r} p_{rs})b_s =$$

$$\sum_{0}^{\infty}{}_{r} p'_r b_r + \sum_{0}^{\infty}{}_{s} p''_s b_s \quad .$$

Consider the marginal distributions $\pi'(1) := \{\pi'_{ri}=p'_r q'_{ri}, \ r \in N, \ i \in N\}$ and $\pi''(1) := \{\pi''_{sj}=p''_s \ q''_{sj}, \ s \in N, \ j \in N\}$ of a given probability distribution $\pi(2) := \{\pi_{risj}=p_{rs}q_{rsij}, \ r \in N, \ s \in N, \ i \in N, \ j \in N\}$. Let m'_r be the total number of $\{q'_{ri}, \ i \in N\}$ that are greater than 0, and m''_s the total number of $\{q''_{sj}, \ j \in N\}$ that are greater than 0, let m_{rs} be defined as in (2.3).

For each $r, s \in N$ s.t $p_{rs} \neq 0$, it is easy to see that

$$m_{rs} \leq m'_r\, m''_s \quad .$$

Hence for every $r, s \in N$ we have

$$p_{rs} \log m_{rs} \leq p_{rs} \log m'_r + p_{rs} \log m''_s$$

and with $c \geq 0$

$$c \sum_0^\infty {}_r \sum_0^\infty {}_s\, p_{rs} \log m_{rs} \leq c \sum_0^\infty {}_r \sum_0^\infty {}_s\, p_{rs} \log m'_r + c \sum_0^\infty {}_r \sum_0^\infty {}_s\, p_{rs} \log m''_s$$

$$= c \sum_0^\infty {}_r\, p'_r \log m'_r + c \sum_0^\infty {}_s\, p''_s \log m''_s \quad .$$

As it concerns the last term on the rhs of (2.3) we have just to make use of the inequality $\log x \leq x - 1$, for all $x > 0$. Indeed with $d \geq 0$, we can write

$$- d \sum_0^\infty {}_r \sum_0^\infty {}_s\, p_{rs} \log p_{rs} + d \sum_0^\infty {}_r\, p'_r \log p'_r + d \sum_0^\infty {}_s\, p''_s \log p''_s =$$

$$d \sum_0^{\infty *} {}_{rs}\, p_{rs} \log \frac{p'_r p''_s}{p_{rs}} \leq d \sum_0^\infty {}_r \sum_0^\infty {}_s\, (p'_r p''_s - p_{rs}) = 0 \quad ,$$

where $\sum_0^{\infty *} {}_{rs}$ denotes the sum over the set of indices

$\{ r, s \in N \mid p_{rs} \neq 0 \}$.

Hence the result.

We introduce now a Lemma that will play an important role in the proof of the representation theorem both in the discrete and in the continuous case.

Let $W_0(2)$ be the set of all probability distributions

$P(2) := \{ p_{rs}, r \in N, s \in N \mid p_{rs} \geq 0 \ \forall r,s \in N, \ \sum_0^\infty r \sum_0^\infty p_{rs} = 1,$

$p_{rs} = 0$ *for* r,s *sufficiently large* $\}$, *and* $W_0(1)$ *the set of all probability distributions* $P(1) := \{ p_r, r \in N \mid p_r \geq 0 \ \forall r \in N,$

$\sum_0^\infty r \, p_r = 1, \ p_r = 0$ *for* r *sufficiently large* $\}$.

If a probability distribution $P^*(2) \in W_0(2)$ *is such that* $p^*_{rs} = 0$ *for each* $r \in N$ *and all* $s > 0$ (*or* $p^*_{rs} = 0$ *for each* $s \in N$ *and all* $r > 0$) *we identify* $P(2)$ *to its marginal distribution* $P^{*\prime} := \{p^{*\prime}_r = p^*_{r0}, r \in N\}$ $(P^{*\prime\prime} := \{ p^{*\prime\prime}_s = p^*_{0s}, s \in N \})$.

<u>*Lemma*</u>.- *If a mapping* $K : W_0(2) \to R$ *has the properties*

(a) *for every* $P(2) \in W_0(2)$ *with marginal distributions* $P'(1)$ *and* $P''(1)$

$$K(P(2)) \leq K_1(P'(1)) + K_1(P''(1)) \qquad (\textit{subadditivity})$$

where K_1 *denotes the restriction of* K *to* $W_0(1)$,

(b) *if* $P(2) = P'(1)P''(1)$, *i.e if* $p_{rs} = p'_r p''_s \ \forall r,s \in N$, *where again* $P'(1)$ *and* $P''(1)$ *are the marginal distributions of* $P(2)$, *then*

$$K(P(2)) = K_1(P'(1)) + K_1(P''(1)) \qquad (\textit{additivity}),$$

(c) *the exchange of any* p_{rs} *with* p_{sr} *does not affect* K(*i.e. for each* $(r,s) \in N \times N$, $(r,s) \to K$ *is symmetric*),

(d) *let* $P_{x,m,n}(1) := \{p_r, r \in N\}$, $x \in [0,1]$, $0 \leq m < n$, *be such that* $p_r = 0$ *for all* $r \neq m,n$, $p_m = 1-x$, $p_n = x$; *the function* $x \to K_1(P_{x,m,n}(1))$, $x \in |0,1|$, *is continuous at* 0, then

$$(2.9) \quad K_1(P(1)) = \sum_0^\infty {}_n\, p_n b_n - d \sum_0^\infty {}_n\, p_n \log p_n \quad \forall \quad P(1) \in W_0(1),$$

where b_n, $n=0,1\ldots$, and d are real numbers, $b_0 = 0$, $d \geq 0$.

Proof: As in ref. 3, the proof is essentially based on a fundamental set of inequalities that can be derived from properties (a), (b) and (c). Here the proof is obviously more complicated than that in ref. 3, since $P(1) \to K_1$ is not assumed to be symmetric.

We start by considering a probability distribution $P'_{x,m,n}(1) :=$ $\{ p'_r,\ r \in N \mid p'_r = 0 \ \forall\ r > n,\ p'_m = p(1-x),\ p'_n = px,\ p \in [0,1], x \in [0,1]$, with $m,n \in N$ fixed, $m < n\}$, and the probability distribution $P''_{y,m,n}(1) := \{\bar{p}''_r,\ r \in N \mid \bar{p}''_r = 0 \ \forall\ r \neq m,n, \bar{p}''_m = 1-y, \bar{p}''_n = y$, $y \in [0,1]\ \}$.

Let

$$f_{m,n}(y) := K_1(\bar{P}''_{y,m,n}(1)) \quad ;$$

because of (b) we have

$$K_1(P'_{x,m,n}(1)) + K_1(\bar{P}''_{y,m,n}(1)) = K_1(P'_{x,m,n}(1)) + f_{m,n}(y)$$

$$= K(P_{x,y,m,n}(2))$$

where $P_{x,y,m,n}(2) := \{p_{rs},\ r \in N,\ s \in N \mid p_{mm} = p(1-x)(1-y),\ p_{mn} =$ $p(1-x)y,\ p_{nm} = px(1-y),\ p_{nn} = pxy,\ p_{rm} = p'_r(1-y) \ \forall\ r \neq m,n,\ p_{rn} =$ $p'_r y \ \forall\ r \neq m,n,\ p_{rs} = 0$ for all $r, s \in N, s \neq m,n$. Now we interchange p_{mn} and p_{nm} in $P_{x,y,m,n}(2)$; let $P^T_{x,y,m,n}(2)$ be the resulting probability distribution.

Because of (c) : $K(P_{x,y,m,n}(2)) = K(P^T_{x,y,m,n}(2))$. Hence

$$K(P'_{x,m,n}(1)) + f_{m,n}(y) = K(P^T_{x,t,m,n}(2)) ,$$

and by recourse to (a)

$$K_1(P'_{x,m,n}(1)) + f_{m,n}(y) \leq K_1(P'_{y,m,n}(1)) + f_{m,n}(px+ (1-p)y) ,$$

where $P'_{y,m,n}(1)$ denotes the probability distribution $P'_{x,m,n}(1)$ - when we replace x with y. From the last inequality we have

(2.10) $K_1(P'_{x,m,n}(1)) - K_1(P'_{y,m,n}(1)) \leq f_{m,n}(px+(1-p)y) - f_{m,n}(y)$,

for every $p,x,y \in [0,1]$.

By interchanging x and y we get

(2.11) $K_1(P'_{y,m,n}(1)) - K_1(P'_{x,m,n}(1)) \leq f_{m,n}(py+(1-p)x) - f_{m,n}(x)$,

and by comparing (2.10) and (2.11) we have

(2.12) $f_{m,n}(x) - f_{m,n}(py+(1-p)x) \leq K_1(P'_{x,m,n}(1)) - K_1(P'_{y,m,n}(1))$

$\leq f_{m,n}(px+ (1-p)y) - f_{m,n}(y)$,

for all $p,x,y \in [0,1]$ and all $m,n \in N$, $m < n$.

We shall now prove some properties of the functions $f_{m,n}$, $m,n \in N$.

Namely

(i) $f_{on}(0) = f_{o1}(0) = 0$;

(ii) f_{mn} is continuous on $[0,1[$;

(iii) f_{mn} is concave on $[0,1]$, that is

$f_{mn}((1-\lambda)x+ \lambda y) \geq (1-\lambda) f_{mn}(x) + \lambda f_{mn}(y)$ for all $\lambda, x, y \in [0,1]$;

(iv) the right and the left derivatives $D^+ f_{mn}$ and $D^- f_{mn}$ exist everywhere on $[0,1[$ and $]0,1]$ respectively;

(v) $D^+ f_{mn}$ and $D^- f_{mn}$ are finite on $]0,1[$.

For the proof of (iii), (iv) and (v) we refer directly to ref. 3 (page 137) since they are consequences of (2.12) only, and (2.12) is exactly the same as (24) in ref. 3.

Property (i) is a consequence of property (b); indeed because of (b) we can write

$$f_{01}(0) + f_{01}(0) = K(P_0(2))$$

with $P_0(2) := \{p_{rs},\ r \in N,\ s \in N \mid p_{00}=1,\ p_{rs}=0$ for all r or s greater than $0\}$. But, with $P_0'(1) := \{p_r' \mid p_0' = 1,\ p_r' = 0\ \forall r > 0\}$: $K(P_0'(1)) = f_{01}(0) := K(P_0(2))$. Hence

$$f_{01}(0) + f_{01}(0) = f_{01}(0) ,$$

and this implies $f_{01}(0) = 0$.

The functions $f_{m,n}$ are continuous at 0 by assumption (property (d)).

Because of (v), the functions $f_{m,n}$ are right-continuous and left-continuous at each $x \in]0,1[$. Thus they can have at most jump discontinuities on $]0,1[$, but this is ruled out by (iii). Therefore the functions $f_{m,n}$ are continuous on $[0,1[$.

At this point by recourse to (2.12) and using the same arguments as in ref. 3 it is easy to prove that for each $m,n \in N$, $n > m$, the -function

$$x \rightarrow K_1'(P'_{x,m,n}(1)) - p\, f_{m,n}(x)$$

is differentiable and its derivative is 0 on $]0,1[$. Thus this -function is constant on $]0,1[$, the constant depending on $m,n,p_0',\ldots,$ $p_{m-1}', p_{m+1}', \ldots, p_{n-2}', p$. Thus

(2.13) $K_1(P'_{x,m,n}(1)) = p\, f_{m,n}(x) + J_{m,n}(p'_0,\dots,p'_{m-1},p'_{m+1},\dots,p'_{n-2},p)$,

for all $x \in]0,1[$, $m, n \in N$, $n > m$.

It is also easy to see that by (2.12) the function $x \to K_1(P'_{x,m,n}(1))$ is continuous at 0. So (2.13) holds for $x=0$ too, that gives

(2.14) $J_{m,n}(p'_0,\dots,p'_{m-1},p'_{m+1},\dots,p'_{n-2},p) = K_1(P'_{0,m,n}(1)) - p\, f_{m,n}(0)$.

Thus we can rewrite (2.13) in the following form

(2.15) $K_1(P'_{x,m,n}(1)) = p\,[f_{m,n}(x) - f_{m,n}(0)] + K_1(P'_{0,m,n}(1))$

for all $x \in [0,1[$, $m, n \in N$, $n > m$.

Let $g_n(x) := f_{n-1,n}(x) - f_{n-1,n}(0)$ for each $n \geq 1$, $x \in [0,1]$; by (2.15) for $m = n-1$, we have

(2.16) $K_1(P'_{x,n-1,n}(1)) = p\, g_n(x) + K_1(P'_{0,n-1,n}(1))$

for all $x \in [0,1[$, $n \geq 1$.

Now take any probability distribution $P_n(1) := \{p_r,\ r \in N \mid p_r = 0\ \forall r \geq n,\ p_r > 0\ \forall r < n\}$ and for each $s, n \in N$, $s < n$, let $P_{s,n}(1) := \{\bar{p}_r,\ r \in N \mid \bar{p}_r = p_r \in P_n(1)\ \forall r < s, \bar{p}_s = \sum_s^n r\ p_r,\ \bar{p}_r = 0\ \forall r > s\}$.

Using (2.16) with $x = p_n/(p_{n-1} + p_n)$ we get

(2.17) $$K_1(P_n(1)) = (p_{n-1} + p_n)\, g_n\left(\frac{p_n}{p_{n-1} + p_n}\right) + K_1(P_{n-1,n}(1))$$

and by recursivity and taking into account that $K_1(P_{0,n}(1)) = 0$ by (i) we have

(2.18) $$K_1(P_n(1)) = \sum_1^n r \left(\sum_{r-1}^n i\ p_i\right) g_r\left(\frac{\bar{p}_r}{\sum_{r-1}^n j\ p_j}\right).$$

Let $h_n(x) := f_{n-2,n}(x) - f_{n-2,n}(0)$, $n \in N$, $n \geq 2$, $x \in [0,1]$. By a further recourse to (2.15) for $m = n-2$ and with $x = p_n/(p_{n-2,n} + p_n)$, we obtain

$$K_1(P_n(1)) = (p_{n-2} + p_n)\, h_n\left(\frac{p_n}{p_{n-2} + p_n}\right) + K_1(P^*_{n-1}(1)) \quad ,$$

where $P^*_{n-1}(1) := \{ p^*_r,\ r \in N \mid p^*_r = p_r \ \forall\ r \neq n-2,\ p^*_{n-2} = p_{n-2} + p_n\}$, and by recourse to (2.17)

$$K_1(P^*_{n-1}(1)) = (p_{n-2}+p_{n-1}+p_n)\, g_{n-1}\left(\frac{p_{n-1}}{p_{n-2} + p_{n-1} + p_n}\right) + K_1(P_{n-2,n}(1)).$$

Thus

$$K_1(P_n(1)) = (p_{n-2} + p_n)\, h_n\left(\frac{p_n}{p_{n-2} + p_n}\right)$$

$$+ (p_{n-2}+p_{n-1}+p_n)\, g_{n-1}\left(\frac{p_{n-1}}{p_{n-2}+p_{n-1}+p_n}\right) + K_1(P_{n-2,n}(1)).$$

But by (2.17) we have also

$$K_1(P_n(1)) = (p_{n-1} + p_n)\, g_n\left(\frac{p_n}{p_{n-1} + p_n}\right)$$

$$+ (p_{n-2}+p_{n-1}+p_n)\, g_{n-1}\left(\frac{p_{n-1}+p_n}{p_{n-2}+p_{n-1}+p_n}\right) + K_1(P_{n-2,n}(1)),$$

hence

$$(2.19) \quad (p_{n-2}+p_n)\, h_n\left(\frac{p_n}{p_{n-2}+p_n}\right) + (p_{n-2}+p_{n-1}+p_n) g_{n-1}\left(\frac{p_{n-1}}{p_{n-2} + p_{n-1} + p_n}\right)$$

$$= (p_{n-1}+p_n)\, g_n\left(\frac{p_n}{p_{n-1}+p_n}\right) + (p_{n-2}+p_{n-1}+p_n) g_{n-1}\left(\frac{p_{n-1} + p_n}{p_{n-2}+p_{n-1} + p_n}\right) \quad ,$$

for all $n \geq 2$, $p_{n-2} \in]0,1[$, $p_{n-1} \in]0,1[$, and because of (ii) and (d) for all $p_n \in [0,1[$. By setting $x := p_{n-2}$, $y := p_{n-1}$ and $z := p_n$, (2.19) leads to the functional equation

$$(2.20) \quad (x+z)\, h_n\left(\frac{z}{x+z}\right) + (x+y+z)\, g_{n-1}\left(\frac{y}{x+y+z}\right)$$

$$= (y+z)\, g_n\left(\frac{z}{y+z}\right) + (x+y+z)\, g_{n-1}\left(\frac{y+z}{x+y+z}\right)$$

for all $n \geq 2$, $x, y \in]0,1[$, $z \in [0,1[$, $x + y + z \leq 1$.

Since by (ii), $h_n (n \geq 2)$ and $g_n (n \geq 0)$ are all continuous on $[0,1[$, from (2,20) we can see that $g_n (n \geq 2)$ is continuous at 1, and

$$(x+z)\, h_n\left(\frac{z}{x+z}\right) = z g_n(1) + (x+z)\, g_{n-1}\left(\frac{z}{x+z}\right)$$

holds.

Substituting into (2.20),

$$(2.21) \quad z\, g_n(1) + (x+z)\, g_{n-1}\left(\frac{z}{x+z}\right) + (x+y+z)\, g_{n-1}\left(\frac{y}{x+y+z}\right)$$

$$= (y+z)\, g_n\left(\frac{z}{y+z}\right) + (x+y+z)\, g_{n-1}\left(\frac{y+z}{x+y+z}\right).$$

Since g_n is continuous at 1 for all $n \geq 2$, by taking the limit of both sides of (2.21) as $x \to 0$, we have

$$z\, g_n(1) + z\, g_{n-1}(1) + (y+z)\, g_{n-1}\left(\frac{y}{y+z}\right)$$

$$= (y+z)\, g_n\left(\frac{z}{y+z}\right) + (y+z)\, g_{n-1}(1),$$

for all $n \geq 3$. Thus

(2.22) $$(y+z)\, g_n\left(\frac{z}{y+z}\right) = (y+z)\, g_{n-1}\left(\frac{y}{y+z}\right) + z\, g_n(1) - y\, g_{n-1}(1).$$

By using (2.22) we can rewrite (2.21) as

(2.23) $$y\, g_{n-1}(1) + (x+z)\, g_{n-1}\left(\frac{z}{x+z}\right) + (x+y+z)\, g_{n-1}\left(\frac{y}{x+y+z}\right)$$

$$= (y+z)\, g_{n-1}\left(\frac{y}{y+z}\right) + (x+y+z)\, g_{n-1}\left(\frac{y+z}{x+y+z}\right)$$

for all $n \geq 3$.

By the substitution

(2.24) $$g_{n-1}(x) - x\, g_{n-1}(0) =: \psi_n(x) \quad ,$$

equation (2.23) yields

(2.25) $$(x+z)\, \psi_n\left(\frac{z}{x+z}\right) + (x+y+z)\, \psi_n\left(\frac{y}{x+y+z}\right)$$

$$= (y+z)\, \psi_n\left(\frac{y}{y+z}\right) + (x+y+z)\, \psi_n\left(\frac{y+z}{x+y+z}\right),$$

$\forall\, n \geq 3,\ x, y \in\,]0,1[,\ z \in [0,1[,\ x+y+z \leq 1$ and ψ_n continuous $[0,1]$.

For $x+y+z = 1$, eq. (2.25) reduces to

(2.26) $$(1-y)\, \psi_n\left(\frac{z}{1-y}\right) + \psi_n(y) = (y+z)\, \psi_n\left(\frac{y}{y+z}\right) + \psi_n(y+z),$$

$\forall\, n \geq 3,\ y \in\,]0,1[,\ z \in [0,1[,\ y+z \leq 1.$

Following the sume classical procedure as in ref. 3 (page 141), it is easy to derive the general continuous solution of (2.26). It is given by

$$\psi_n(x) = A_n\, [-(1-x)\log(1-x) - x\log x] \quad , x \in [0,1]$$

where because of (iii) $A_n \geq 0$. Then, by (2.24) we have

$$g_n(x) = A_n\, [-(1-x)\log(1-x) - x\log x] + b_n x \quad \forall\, n \geq 2,$$

with $\bar{b}_n := g_n(1)$, and by substituting into (2.22) we recognize - that $A_n = A_{n-1} = d \geq 0$, for all $n \geq 3$. Thus

(2.27) $g_n(x) = -d\,[(1-x)\log(1-x) + x\log x] + \bar{b}_n x \;\; \forall\, n \geq 2,\; x \in [0,1]$.

For $n=2$, $x+y+z = 1$, (2.21) leads to

$$(1-y)\, g_1\left(\frac{z}{1-y}\right) + g_1(y) - g_1(y+z) = -d\left(y \log \frac{y}{y+z} + z \log \frac{z}{y+z}\right)$$

$\forall\, y \in\,]0,1[,\; z \in [0,1[,\; y+z \leq 1$. It is easy to verify that - the general continuous solution of this functional equation is gi ven by

$$g_1(x) = -d\,[(1-x)\log(1-x) + x\log x] + \bar{b}_1 x\,,$$

where $\bar{b}_1 = g_1(1)$. Hence (2.27) holds for all $n \geq 1$. So we can rewrite (2.18) in the following form

$$K_1(P_n(1)) = \sum_1^n r\, p_r \sum_1^r i\, \bar{b}_i - d \sum_0^n r\, p_r \log p_r \quad \forall\; n \geq 1.$$

by setting $b_r = \sum_1^r i\, \bar{b}_i$ and using property (d) and inequalities (2.12) which imply the continuity of $x \to K_1(P'_{x,m,n}(1))$ on $[0,1]$ for each $m, n \in N$, $m < n$, we have

$$K_1(P(1)) = \sum_0^\infty n\, p_n\, b_n - d \sum_0^\infty n\, p_n \log p_n \quad \forall\; P(1) \in W_0(1),$$

where b_n, $n=0,1,\ldots$, and d are real numbers, $b_0 = 0$, $d \geq 0$. - Hence the result.

3.- The representation theorem.

We are now ready to prove the following theorem.

Theorem.- If a mapping $H : W(2) \to R$ has the properties (1), (2), (3a), (3b), (4) then its restriction to $W(1)$ has the form

$$H_1(\pi(1)) = -a \sum_0^\infty{}_n p_n \sum_0^\infty{}_i q_{ni} \log q_{ni} + \sum_0^\infty{}_n p_n b_n$$

$$+ c \sum_0^\infty{}_n p_n \log m_n - d \sum_0^\infty{}_n p_n \log p_n \quad \forall \pi(1) \in W(1),$$

where a, b_n $(n=0,1,\ldots,)$, c and d are real numbers, $a \geq 0$, $b_0 = 0$, $c \geq 0$, $d \geq 0$; m_n denotes the total number of $\{q_{ni}, i \in N\}$ that are greater than 0.

Proof: We introduce the following notations

$$\bar{\pi}_{m,n}(2) := \{ \pi_{risj}=p_{rs}q_{rsij},\ r \in N,\ s \in N,\ i \in N,\ j \in N \mid p_{rs} = 0$$
$$\forall (r,s) \neq (m,n)\} \in W(2),\ (m,n) \in N\times N$$

$$\bar{\pi}_n(1) := \{ \pi_{ri}=p_r q_{ri},\ r \in N,\ i \in N \mid p_r=0 \ \forall r\neq n\} \in W(1),\ n \in N,$$

$$Q_{m,n,o}(2) := \{ \pi_{risj}=p_{rs}q_{rsij},\ r \in N,\ s \in N,\ i \in N,\ j \in N \mid p_{rs}=0$$
$$\forall (r,s) \neq (m,n),\ q_{mnoo}=1,\ q_{mnij}=0 \text{ if } i > 0 \text{ or } j > 0\} \in W(2),$$
$$(m,n) \in N\times N,$$

$$Q_{n,o}(1) := \{ \pi_{ri}=p_r q_{ri},\ r \in N,\ i \in N \mid p_r=0 \ \forall r \neq n,\ q_{n,o} = 1,$$
$$q_{ni} = 0 \ \forall i > 0\} \in W(1),\ n \in N.$$

We shall denote with $\bar{W}(2)$ and $\bar{W}(1)$ the set of all such probability distributions $\bar{\pi}_{m,n}(2)$ and $\bar{\pi}_n(1)$, respectively, $m,n = 0,1,\ldots$. For each $\bar{\pi}_{m,n}(2)$ in $\bar{W}(2)$ we can always find a probability distribution $\bar{\pi}^*_{m,n}(2) \in \bar{W}(2)$ by just reordering the probabilities q_{mnij} in $\bar{\pi}_{m,n}(2)$ so that $Q_{n,o}(1)$ is one of the two marginal - distributions of $\bar{\pi}^*_{m,n}(2)$ and $\bar{\pi}^*_{m,n}(2) = \bar{\pi}'_m(1) \times Q_{n,o}(1)$, where $\bar{\pi}'_m(1)$ denotes the other marginal distribution of $\bar{\pi}^*_{m,n}(2)$ and x means the product probability distribution of $\bar{\pi}'_m(1)$ and $Q_{n,o}(1)$. By using properties (2) and (3a) we have

$$(3.1) \quad H(\bar{\pi}_{m,n}(2)) = H(\bar{\pi}^*_{m,n}(2)) = H_1(\bar{\pi}'_m(1)) + H_1(Q_{n,o}(1)).$$

Set $b_n := H_1(Q_{n,o}(1))$ and define

$$I(\bar{\pi}_{m,n}(2)) := H(\bar{\pi}_{m,n}(2)) - b_m - b_n, \ \forall \ \bar{\pi}_{m,n}(2) \in \bar{W}(2).$$

Note that because of (i) $b_o = 0$. It can be easily checked that the function I does not depend on m and n but only on the entries of the probability distribution $\bar{\pi}_{m,n}(2)$ and that I coincides with its restriction $I_1(\bar{\pi}_n(1)) = H_1(\bar{\pi}_n(1)) - b_n$ (to $\bar{W}(1)$). Moreover I is subadditive, additive, expansible and symmetric; hence, by recourse to the theorem in ref. 3, we have

$$(3.2) \quad I(\bar{\pi}_n(1)) = -a \sum_{o}^{\infty}{}_i \, q_{ni} \log q_{ni} + c \log m_n \quad \forall \ \bar{\pi}_n \in \bar{W}(1),$$

where a and c are real numbers, $a \geq 0$. Thus

$$(3.3) \quad H_1(\bar{\pi}_n(1)) = -a \sum_{o}^{\infty}{}_i \, q_{ni} \log q_{ni} + c \log m_n + b_n \quad \forall \ \bar{\pi}_n \in \bar{W}(1).$$

We consider now any two probability distributions $\vec{x} = \{x_i,\ i \in N\}$ and $\vec{y} = \{y_j,\ j \in N\}$, with $x_i = 0$ and $y_j = 0$ for i and j, respectively, sufficiently large, and the probability distributions $\pi_{x,n}(1) \in W(1)$ and $\bar{\pi}_{y,n} \in \bar{W}(1)$ defined for any $n \in N,\ n \geq 1$, by

$$\pi_{x,n}(1) :=\{ \pi_{ri} = p_r q_{ri}\ \forall\ r \neq n, \quad \pi_{ni} = p_n x_i,\ i \in N \}$$

$$\bar{\pi}_{y,n}(1) :=\{ \pi_{sj} = 0\ \forall\ s \neq n, \quad \pi_{nj} = y_j,\ j \in N \} \ .$$

By properties (2), (3a) and (1) (in that order) we get

$$H_1(\pi_{x,n}(1)) + H_1(\bar{\pi}_{y,n}(1)) = H(\pi_{x,y,n}(2)) = H(\pi^T_{x,y,n}(2))$$

$$\leq H_1(\pi_{y,n}(1)) + H_1(\bar{\pi}_{p_n x+(1-p_n)y,\ n}(1)) \ ,$$

where:

$$\pi_{x,y,n}(2) = \pi_{x,n}(1)\times\bar{\pi}_{y,n}(1) :=\{\pi^*_{risj}=0\ \forall\ r,i,s,j \in N,\ \ s \neq n,$$

$$\pi^*_{rinj} = p_r q_{ri} y_j\ \forall\ r,i,j \in N,\ r \neq n, \quad \pi^*_{ninj} = p_n x_i y_j$$

$$\forall\ i,j \in N \},$$

$$\pi^T_{x,y,n}(2) :=\{ \pi^T_{risj} = \pi^*_{risj}\ \forall\ r,i,s,j \in N,\ r \neq n \ \text{ or } \ s \neq n.$$

$$\pi^T_{ninj} = \pi^*_{njni} = p_n x_j y_i \quad \forall\ i,\ j \in N \} \ ,$$

$$\pi_{y,n}(1) :=\{ \pi^T_{ri} = \sum_{0}^{\infty}{}_s \sum_{0}^{\infty}{}_j\ \pi^T_{risj},\ \forall\ r,i \in N, \text{ i.e } \pi^T_{ri} = p_r q_{ri}$$

$$\forall\ r,\ i \in N,\ r \neq n, \quad \pi^T_{ni} = p_n y_i\ \forall\ i \in N \},$$

$$\bar{\pi}_{p_n x+(1-p_n)\,y,n}(1):=\{\pi^T_{s,j}=\sum_0^\infty{}_r\sum_0^\infty{}_i\,\pi^T_{risj}\ \forall\, s,j\in N,\ i.e.\ \pi^T_{sj}=0$$

$$\forall\, s,j\in N,\ s\neq n,\ \pi^T_{nj}=p_n x_j+(1-p_n)y_j\ \ \forall\, j\in N\}.$$

By interchanging x and y and taking (3.3) into consideration, we get

$$(3.4)\quad -a\sum_0^\infty{}_i\, x_i \log x_i + a\sum_0^\infty{}_i((1-p_n)x_i+p_n y_i)\log((1-p_n)x_i+p_n y_i)$$

$$\leq H_1(\pi_{x,n}(1)) - H_1(\pi_{y,n}(1))$$

$$\leq -a\sum_0^\infty{}_i(p_n x_i+(1-p_n)y_i)\log(p_n x_i+(1-p_n)y_i)+a\sum_0^\infty{}_i y_i\log y_i$$

for each $n\geq 1$, $p_n\ \varepsilon\ [0,1]$, and every $\vec{x}$ and $\vec{y}$ such that $y_i=0$ iff $x_i=0$, $\{i=0,1,\ldots,\}$.

Inequalities (3.4) imply almost immediately that the function

$$\vec{x}\rightarrow H_1(\pi_{x,n}(1)) + a\,p_n\sum_0^\infty{}_i\, x_i\log x_i$$

is constant on the set Γ'_{m_n} of all probability distributions $\vec{x}=\{x_0, x_1, \ldots\}$ having the same number m_n of nonzero probabilities.

Hence for any $\pi(1)\ \varepsilon\ W(1)$ we can write

$$(3.5)\quad H_1(\pi(1)) = -a\sum_0^\infty{}_r\, p_r\sum_0^\infty{}_i\, q_{ri}\log q_{ri} + A(\vec{m},\vec{p}),$$

where $A(\vec{m},\vec{p})$ is a function of the sequence of integers

$\vec{m} = \{m_0 = 1,\ m_1,\ m_2,\ \ldots\}$ and the probability distribution $\vec{p} = \{p_0,\ p_1,\ p_2,\ \ldots\}$.

To find the representation of $A(\vec{m},\vec{p})$, with a fixed $\vec{m} = \{m_0 = 1,\ m_1,\ m_2,\ \ldots\}$, we consider the following probability distributions

$\pi_u(2) := \{ \pi_{risj} = p_{rs}q_{rsij},\ r,s,i,j \in N \mid q_{rsij} = \frac{1}{m_r m_s}$ for all $i = 0,1,\ldots,m_r-1,\ j = 0,1,\ldots,m_s-1,\ q_{rsij} = 0$ for each $r,s \in N$ and all $i \geq m_r$ and each $r,s \in N$ and all $j \geq m_s \} \in W(2)$ and their marginals $\pi'_u(1)$ and $\pi''_u(1)$. We shall denote with $W_u(2)$ the set of all the probability distributions $\pi_u(2)$ and with $W_u(1)$ the set of their marginals.

It can be easily seen that the mappings

$$K : P(2) \to H(\pi_u(2)) \qquad [\, P(2) \in W_0(2) \,],$$

$$K_1 : P(1) \to H_1(\pi_u(1)) \qquad [\, P(1) \in W_0(1) \,]$$

verify all the assumptions of the Lemma in section 2. Thus we have

$$(3.6)\quad H_1(\pi_u(1)) = \sum_{0}^{\infty}{}_r\, p_r b^*_r(\vec{m}) - d \sum_{0}^{\infty}{}_r\, p_r \log p_r \quad \forall\ \pi_u(1) \in W_u(1),$$

where $b^*_r(\vec{m})$, $r = 0,1,\ldots,$ are real valued functions of $\vec{m}$, $b_0(\vec{m}) = 0$, and d is a nonnegative real constant.

On the other hand Eq. (3.5) implies

$$(3.7)\quad H_1(\pi_u(1)) = a \sum_{0}^{\infty}{}_r\, p_r \log m_r + A(\vec{m},\vec{p}) \quad ;$$

thus, by (3.6) and (3.7),

$$(3.8)\quad A(\vec{m},\vec{p}) = \sum_{0}^{\infty}{}_r\, p_r(b_r^*(\vec{m}) - a \log m_r) - d \sum_{0}^{\infty}{}_r\, p_r \log p_r \;;$$

since Eq. (3.8) holds for every probability distribution $\vec{p}=\{p_0,p_1,p_2,\dots\}$ in particular for $\vec{p}=\vec{p}_{o,n} := \{p_r,\ r \in N\ \ p_r=0\ \ r \neq n,\ p_n = 1\}$, we have

$$(3.9)\quad A(\vec{m},\vec{p}_{o,n}) = b_n^*(\vec{m}) - a \log m_n \qquad (n \in N).$$

But by (3.3) and (3.7) we also have

$$(3.10)\quad A(\vec{m},\vec{p}_{o,n}) = c \log m_n + b_n \quad ,$$

thus we get

$$(3.11)\quad b_n^*(m) = a \log m_n + c \log m_n + b_n \quad \forall\ n \in N,\ \ n > 0.$$

Finally, by (3.5), (3.8) and (3.11) we obtain

$$H_1(\pi(1)) = - a \sum_{0}^{\infty}{}_r\, p_r \sum_{0}^{\infty}{}_i\, q_{ri} \log q_{ri} + c \sum_{0}^{\infty}{}_r\, p_r \log m_r$$

$$+ \sum_{0}^{\infty}{}_r\, p_r b_r - d \sum_{0}^{\infty}{}_r\, p_r \log p_r, \quad \forall\ \pi(1) \in W(1) .$$

Hence the result.

<u>Corollary</u>.- The mean entropy (1.3) is the only entropy which satisfies (1), (2), (3a), (3b), (4) and

$$(5)\quad S_1(\pi_{1,o,n+h}(1)) - S_1(\pi_{1,o,n}(1)) = S_1(\pi_{1,o,h}(1)) - S_1(\pi_{o,o,1}(1))$$

for all $n,h \in N$, $n > 0$, $h > 0$.

<u>Proof</u>: By the theorem we have

$$S_1(\pi_{1,0,r}(1)) = H_1(\pi_{1,0,r}(1)) = b_r \quad \forall \quad r \in N, \quad r > 0$$

$$S_1(\pi_{0,0,1}(1)) = H_1(\pi_{0,0,1}(1)) = b_0 = 0 \quad ,$$

By setting $H_1(\pi_{1,0,1}(1)) = b_1 =: b$, *and using* (5) *we get*

$$b_{n+1} = b_n + b_1 = b_n + b = (n+1)b$$

and this concludes the proof.

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References.

1. D. W. Robinson and D. Ruelle (1967), Mean Entropy of States in Classical Statistical Mechanics. Commun. Math. Phys. 5, 288-300.

2. E. H. Lieb (1975), Some Convexity and Subadditivity Properties of Entropy. Bull. Amer. Math. Soc. 81, n. 1, 1-13.

3. J. Aczél, B. Forte and C. T. Ng (1974), Why the Shannon and Hartley Entropies are 'Natural'. Adv. Appl. Prob. 6, 131-146.

4. B. Forte and C.C.A. Sastri (1975), Is something missing in the Boltzmann Entropy? Journal of Math. Phys. 16, n. 7, - 1453-1456.