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Hydrodynamics in Class B Warped Spacetimes

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We discuss certain general features of type B warped spacetimes which have important consequences on the material content they may admit and its associated dynamics. We show that, for Warped B spacetimes, if shear and anisotropy are nonvanishing, they have to be proportional. We also study some of the physics related to the warping factor and of the underlying decomposable metric. Finally we explore the only possible cases compatible with a type B Warped geometry which satisfy the dominant energy conditions. As an example of the above mentioned consequences we consider a radiating fluid and two non-spherically symmetric metrics which depend upon an arbitrary parameter a, such that for $a = 0$ spherical symmetry is recovered in both cases.

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I. INTRODUCTION.

Given two metric manifolds (M_1, h_1) and (M_2, h_2) and given a smooth real function $\theta : M_1 \to \mathbb{R}$, (*warping function*), one can build a new metric manifold (M, g) by setting $M = M_1 \times M_2$ and

$$g = \pi_1^* h_1 \otimes \mathrm{e}^{2\theta} \pi_2^* h_2,$$

where π_1, π_2 above are the canonical projections onto M_1 and M_2 respectively, such an structure is called *warped product manifold*, and in the case in which (M, g) is a spacetime (i.e.: $\dim M = 4$ and g a Lorentz metric) it is called a warped product spacetime (or simply *warped spacetime*). One of the simplest examples of warped spacetime is provided by the Friedman-Robinson-Walker universe. But the warped structure accommodates a large number of metrics in General Relativity, such as Bertotti-Robinson, Robertson-Walker, Schwarzschild, Reissner-Nordstrom, de Sitter, etc. (see [1] and references therein). Also warped spacetimes can be regarded, in some sense, as generalizations of locally decomposable spacetimes in the sense usually meant in general relativity ([2]).

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The importance of warped spacetimes is that their geometry and, as we will presently show, also its physics, is directly related to the properties of their lower-dimensional factors, which are generally easier to study. The warped product construction provides a useful method for studying large classes of spacetimes. If the warping factor is constant the spacetime is decomposable, such as the Bertotti-Robinson spacetime or the Einstein static universe. Warped product spacetimes with non-constant warping factors are much richer and include such well known examples such as all the spherically, plane and hyperbolically symmetric spacetimes (therefore including Schwarzshild solution), Friedmann Robertson Walker cosmologies, all the static spacetimes, etc. (see [3] and references therein).

Anisotropy and shear properties of fluids in General Relativity have been extensively studied. Shearfree and non shearfree spacetimes have been widely considered in the literature (see for example [4]). On the other hand, the assumption of local anisotropy of pressure (i.e. non pascalian fluids where radial and tangential pressures are different, $P_r \neq P_\perp$), has been proven to be very useful in the study of relativistic compact objects. Although the perfect pascalian fluid assumption (i.e. $P_r = P_\perp$) is supported by solid observational and theoretical grounds, an increasing amount of theoretical evidence strongly suggests that, for certain density ranges, a variety of very interesting physical phenomena may take place giving rise to local anisotropy (see [5] and references therein).

The purpose of this paper is twofold, on the one hand we present and discuss in detail certain general features of type B warped spacetimes which have important consequences on the material content they may admit and its associated dynamics; on the other hand, a thorough study of the dissipative anisotropic fluid dynamics in such spacetimes is carried out, with particular emphasis put on a set of geometrical and physical variables which appear to play a special role in the evolution of such systems.

In this paper, it will be shown how the local anisotropy of pressures and the shear of relevant velocity fields are closely related; in fact, and for B_T warped spacetimes, we will show that if both, shear and anisotropy, are nonvanishing they have to be proportional. We shall also show how isotropic and anisotropic physics are related to the warping factor or to the conformally related decomposable metric. Further, we will explore the possible material contents that are compatible with a type B_T warped geometry and satisfy the dominant energy condition. As an example of the above mentioned consequences we shall consider a radiating fluid. Radiative hydrodynamics is a theory of fundamental importance in astrophysics, cosmology, and plasma physics. It has became a very active field with a wide variety of application areas ranging from Plasma laboratory physics, to astrophysical and cosmological scenarios (see the comprehensive treatise of D. Mihalas [6] and his recently updated bibliography [7]).

The mathematical model of radiative hydrodynamics consists of the equations for a two-component medium: matter and radiation, which interact by exchanging energy and momentum; i.e.: anisotropic matter plus radiation (photon/neutrinos) which can be described by the total stress-energy tensor $T_{ab} = T^M_{ab} + T^R_{ab}$ where the material part is described by T^M_{ab}, and $\hat{T}^R_{ab}$ is then the corresponding term for the radiation field. The interaction between matter and radiation is described by a radiative transfer equation through the absorption and emission terms, describing the rate at which matter absorbs and emits photons, and by integral terms describing the scattering of radiation (photons/neutrinos) off matter.

The paper is organized as follows: the first section contains a brief account of the definitions and most immediate properties of warped spacetimes, especially those of the type B, and introduces the notation and conventions used throughout the paper. In Section III some general results regarding the energy-momentum tensor of this class of spacetimes are proven and their implications on the physical content and material dynamics are pointed out relating them to the issues discussed in section II; these results complement and extend those in [8]. Section IV displays some of the consequences of the geometric structure of this type of spacetimes in a simple and, we believe, useful form for the case of a radiating fluid. The restrictions imposed by the energy conditions are explored and illustrated by this simple and useful example. In section **??** we provide a detailed study of the dynamics of dissipative, anisotropic fluids in these geometries, thus generalizing the results by Herrera et al. [9] in the spherically symmetric case. Finally, in section VI we summarize the main results and conclusions.

II. PRELIMINARY RESULTS, NOTATION AND CONVENTIONS.

In this section we set up the notation and summarize some of the results to be used in the remainder of the paper. We recall the basic definitions regarding warped spacetimes and introduce the concepts of adapted observers and adapted tetrads. We also explore the structure of the energy-momentum tensors which are compatible with a type B_T warped geometry.

A. Warped and decomposable spacetimes

As mentioned in the previous section, given two metric manifolds (M_1, h_1) and (M_2, h_2) and a smooth real function $\theta : M_1 \to \mathbb{R}$, (*warping function*), a new metric manifold (*warped product manifold*) (M, g) can be built where $M = M_1 \times M_2$ and

$$g = \pi_1^* h_1 \otimes \mathrm{e}^{2\theta} \pi_2^* h_2, \tag{1}$$

with π_1, π_2 the canonical projections onto M_1 and M_2 respectively (see [10], [11]). Where there is no risk of confusion; we shall omit the projections π_1, π_2 and write from now on:

$$g = h_1 \otimes \mathrm{e}^{2\theta} h_2. \tag{2}$$

Notice that by pulling out the warping factor, we can always rewrite the metric as

$$g = \mathrm{e}^{2\theta} \left(\mathrm{e}^{-2\theta} h_1 \otimes h_2\right) \equiv \mathrm{e}^{2\theta} \left(h_1' \otimes h_2\right) \tag{3}$$

where $h_1' = \mathrm{e}^{-2\theta} h_1$ is also a metric on M_1; thus, a warped manifold is always conformally related to a decomposable one (see [4]).

If $\dim M_1 + \dim M_2 = 4$ and g has Lorentz signature (i.e.: one of the manifolds (M_i, h_i) is Lorentz and the other Riemann), (M, g) is usually referred to as a *warped spacetime*; see [1] and [12] where (local) invariant characterizations are provided along with a classification scheme and a detailed study of the isometries that such spacetimes may admit. If one has either $\dim M_1 = 1$ or $\dim M_2 = 1$, the spacetime is said to be of class A, whereas if $\dim M_1 = \dim M_2 = 2$ it is said to be of class B, which is the class we shall be interested in. Class B is further subdivided into four classes according to the gradient of the warping function: B_T if it is non-null and everywhere tangent to the Lorentz submanifold, B_R if it is null (hence also tangent to the Lorentz submanifold), B_S if it is tangent to the Riemann submanifold, and B_P if it is zero, i.e.: $\theta =$ constant which corresponds to (M, g) being locally decomposable.

Of all the above possibilities we shall only be concerned in this paper with class B_T. Thus, and without loss of generality we shall assume that (M_1, h_1) is Lorentz (coordinates $x^A = (x^0, x^1)$) and (M_2, h_2) is Riemann (coordinates $x^\alpha = (x^2, x^3)$); the warping function θ then being $\theta(x^0, x^1)$. An *adapted* coordinate chart for the whole spacetime manifold M will be denoted as $x^a = (x^A, x^\alpha)$ $a = 0, \ldots, 3$ where x^A and x^α are those defined previously. We shall always use such adapted charts, furthermore and in order to ease out the notation, we shall use the following coordinate names: $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$. At this point, it is worthwhile noticing that spherically, plane and hyperbolically symmetric spacetimes are all special instances of B_T warped spacetimes.

In what follows, we shall write the spacetime metric in the form (3), i.e.: explicitly conformally decomposable, and we shall put $\exp\theta = \omega^{-1}$ for convenience; further, we shall drop primes in (3) as well as the subscripts 1 and 2 in the metrics of the submanifolds M_1 and M_2 where there is no risk of confusion, thus the line element will be written as

$$ds^2 = \omega^{-2}(x^D)\left[h_{AB}(x^D)dx^A dx^B + h_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta\right] \quad (4)$$

i.e.

$$ds^2 \equiv \omega^{-2} d\hat{s}^2 \quad \Leftrightarrow \quad g_{ab} = \omega^{-2}\hat{g}_{ab} \quad (5)$$

where $\hat{g}$ is the underlying conformally related, decomposable metric with line element

$$d\hat{s}^2 = h_{AB}(x^D)dx^A dx^B + h_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta. \quad (6)$$

Since h_{AB} and $h_{\alpha\beta}$ are two two-metrics, one can always choose the coordinates x^A and x^α so that both take diagonal forms (even explicitly conformally flat); thus, and in order to fix our notation further, we shall most often use in our calculations the following form of the metric:

$$ds^2 = \omega^{-2}(x^D)\left[-A^2(t,x)dt^2 + B^2(t,x)dx^2 + e^{2Q(y,z)}(dy^2 + dz^2)\right] \quad (7)$$

We shall denote the covariant derivative with respect to the connection associated with g by a semicolon (or also ∇), whereas that associated with $\hat{g}$ will be noted by a stroke (or alternatively $\hat{\nabla}$); accordingly, tensors defined in $(M, \hat{g})$ or referred to the metric $\hat{g}$ will be noted with a hat '^'.

B. Adapted observers and tetrads

A further important remark concerns observers (congruences of timelike curves) in these spacetimes. A future directed unit timelike vector field $\hat{\vec{v}}$ will be said to be an *adapted observer* in $(M, \hat{g})$ if it is hypersurface orthogonal and everywhere tangent to M_1. These requirements are equivalent to saying that, in an adapted coordinate chart its components are $\hat{v}^a = (\hat{v}^0(x^D), \hat{v}^1(x^D), 0, 0)$. It is easy to see that these observers always exist and that the coordinates x^D may be chosen so that $\hat{v}^1 = 0$ while the metric preserves its diagonal form. We shall construct an *adapted tetrad* in $(M, \hat{g})$ by choosing a unit spacelike vector field $\hat{\vec{p}}$ which is everywhere tangent to M_1 and orthogonal to $\hat{\vec{v}}$; i.e.: $\hat{p}^a = (\hat{p}^0(x^D), \hat{p}^1(x^D), 0, 0)$, and two other unit spacelike vector fields, $\hat{\vec{y}}$, $\hat{\vec{z}}$, which are also hypersurface orthogonal, tangent to M_2 and mutually orthogonal $y^a z_a = 0$ (hence in an adapted chart: $\hat{y}^a = (0, 0, \hat{y}^2(x^\gamma), \hat{y}^3(x^\gamma))$, and something similar for $\hat{z}^a$, also note that $v^a y_a = \ldots = p^a z_a = 0$). In terms of this adapted tetrad one has

$$h_{AB} = -\hat{v}_A\hat{v}_B + \hat{p}_A\hat{p}_B, \quad h_{\alpha\beta} = \hat{y}_\alpha\hat{y}_\beta + \hat{z}_\alpha\hat{z}_\beta. \quad (8)$$

and a trivial calculation now shows that

$$\hat{v}_{A/B} = -a\hat{p}_A\hat{v}_B + \vartheta\hat{p}_A\hat{p}_B, \;\; \hat{v}_{\alpha/\beta} = 0 \quad (9)$$

$$\hat{p}_{A/B} = -a\hat{v}_A\hat{v}_B + \vartheta\hat{v}_A\hat{p}_B, \;\; \hat{p}_{\alpha/\beta} = 0 \quad (10)$$

Notice that ϑ and a are respectively, the expansion and the acceleration of $\hat{\vec{v}}$ in $(M, \hat{g})$. Using the above expressions, the shear associated with $\hat{\vec{v}}$ turns out to be (recall that $\hat{\omega}_{ab} = 0$):

$$\hat{\sigma}_{ab} = \vartheta\left(\hat{p}_a\hat{p}_b - \frac{1}{3}\hat{h}_{ab}\right), \quad \text{with } \hat{h}_{ab} \equiv \hat{g}_{ab} + \hat{v}_a\hat{v}_b, \quad (11)$$

We next define an *adapted observer* in (M, g), $\vec{v}$ to be $\vec{v} = \omega\hat{\vec{v}}$, where $\hat{\vec{v}}$ is any adapted observer in the decomposable spacetime $(M, \hat{g})$ as defined above. Note that $\vec{v}$ is also hypersurface orthogonal and tangent everywhere to M_1, and its components, in any adapted chart, will be functions of the coordinates x^D alone. We construct the rest of an *adapted tetrad* in (M, g) simply as $\vec{p} = \omega\hat{\vec{p}}$, $\vec{y} = \omega\hat{\vec{y}}$, and $\vec{z} = \omega\hat{\vec{z}}$, where the hatted vectors form an adapted tetrad in $(M, \hat{g})$ as defined above. In terms of an adapted tetrad:

$$g_{AB} = -v_A v_B + p_A p_B, \quad g_{\alpha\beta} = y_\alpha y_\beta + z_\alpha z_\beta. \quad (12)$$

Regarding the shear and vorticity of $\vec{v}$ one has [4]:

$$\left.\begin{aligned} \sigma_{ab} &= \omega^{-1}\hat{\sigma}_{ab} = \omega\vartheta\left(p_a p_b - \tfrac{1}{3}h_{ab}\right) \\ h_{ab} &\equiv g_{ab} + v_a v_b, \\ \omega_{ab} &= \omega^{-1}\hat{\omega}_{ab} = 0 \end{aligned}\right\} \quad (13)$$

¿From a geometric point of view, adapted observers and tetrads, seem very natural in both warped and conformally related decomposable spacetimes. As we shall see early on in the next section, they also arise very naturally from physical considerations.

Notice that one could have observers that, while being tangent to M_1 they are not hypersurface orthogonal, e.g.: in the coordinates introduced in (7), consider

$$\hat{\vec{u}} = f\partial_t + B^{-1}[A^2 f^2 - 1]^{1/2}\partial_x \tag{14}$$

where $f = f(x^D, x^\gamma)$ depends on all four coordinates, it is immediate to check that this vector field has non-vanishing vorticity (indeed its components depend on coordinates in both M_1 and M_2 in any adapted chart). We shall briefly return to this point later on, but as already hinted above, such observers are somehow unnatural from a physical viewpoint.

C. Einstein Tensor and Warped Spacetimes

The geometry of the decomposable spacetime $(M, \hat{g})$ imposes certain restrictions that will become important later on in our study of hydrodynamics in warped spacetimes of this class and that have to do with the natural occurrence of the adapted tetrads and observers discussed above.

With the conventions and notation introduced so far, it turns out (see e.g. [13]) that the Einstein tensor in (M, g) can be written as

$$\begin{aligned} G_{ab} &= \hat{G}_{ab} + 2\omega^{-1}\omega_{a/b} \\ &\quad - 2\omega^{-1}\hat{g}^{cd}\left(\omega_{c/d} - \frac{3}{2}\omega^{-1}\omega_c\omega_d\right)\hat{g}_{ab}. \end{aligned} \tag{15}$$

Note that $\hat{R}_{ab}$ is such that

$$\hat{R}_{AB} = \frac{1}{2}R_1 h_{AB}, \quad \hat{R}_{A\alpha} = 0, \quad \hat{R}_{\alpha\beta} = \frac{1}{2}R_2 h_{\alpha\beta}, \tag{16}$$

where R_1 and R_2 are the Ricci scalars associated with the two-metrics h_{AB} and $h_{\alpha\beta}$ respectively. The Ricci scalar $\hat{R}$ is $\hat{R} = R_1 + R_2$, hence

$$\left.\begin{aligned} \hat{G}_{AB} &= -\tfrac{1}{2}R_2 h_{AB}, \\ \hat{G}_{A\beta} &= 0, \\ \hat{G}_{\alpha\beta} &= -\tfrac{1}{2}R_1 h_{\alpha\beta}. \end{aligned}\right\} \tag{17}$$

Furthermore

$$\omega_{A/\alpha} = \omega_{\alpha/A} = 0, \qquad \omega_{\alpha/\beta} = 0, \tag{18}$$

and taking (15) into account, it follows that G_{ab} has box diagonal form:

$$G_{ab} = \begin{pmatrix} G_{AB} & 0 \\ 0 & G_{\alpha\beta} \end{pmatrix}, \tag{19}$$

with

$$\left.\begin{aligned} G_{AB} &= -\tfrac{1}{2}R_2(x^\gamma)h_{AB} + S_{AB}(x^D) \\ G_{A\beta} &= 0 \\ G_{\alpha\beta} &= L(x^D)h_{\alpha\beta} \end{aligned}\right\} \tag{20}$$

where S_{AB} (and therefore G_{AB}) is non-diagonal in the general case.

At this point, it is interesting to notice that, on account of the form of $\hat{G}_{ab}$, it follows that any vector field $\vec{X}$ tangent to M_1 that is an eigenvector of G_{ab} (or equivalently of R_{ab}) will automatically be an eigenvector of $\omega_{a/b}$ and viceversa; and that any vector field $\vec{Y}$ tangent to M_2 that is an eigenvector of G_{ab} (or equivalently of R_{ab}) will also automatically be an eigenvector of $\omega_{a/b}$ and viceversa; in the next section we will show that all eigenvectors of the Einstein tensor are necessarily tangent to M_1 or to M_2, as the block diagonal structure suggests.

Also notice that almost all the physical properties of the spacetime under consideration are somehow encoded in the warping factor ω, since the contribution to the energy momentum tensor $T_{ab} = G_{ab}$ of the underlying decomposable spacetime is simply a shift in the eigenvalues.

We shall dedicate the next section to study the allowed algebraic types of the Einstein tensor, which through Einstein's field equations will provide information on the material content allowed for such spacetimes.

III. MATERIAL CONTENT OF B_T WARPED SPACETIMES.

A. Observers and Matter content

Given a second order symmetric tensor such as the energy-momentum tensor T_{ab} in an arbitrary spacetime (M, g), and given an arbitrary unit timelike vector field $\vec{v}$ (which we shall assume future oriented) defined on M, one can always decompose T_{ab} as follows

$$T_{ab} = \tilde{\rho}v_a v_b + P h_{ab} + \Pi_{ab} + v_a\mathcal{F}_b + \mathcal{F}_a v_b, \tag{21}$$

where h_{ab} is the projector orthogonal to $\vec{v}$, that is: $h_{ab} = g_{ab} + v_a v_b$, and the rest of quantities appearing above are

$$\left.\begin{aligned} &\tilde{\rho} = T_{ab}v^a v^b, \quad P = \tfrac{1}{3}h^{ab}T_{ab} \\ &\mathcal{F}_a = -h_a^{\ c}T_{cd}v^d, \\ &\Pi_{ab} = h_a^{\ c}h_b^{\ d}(T_{cd} - P g_{cd}). \end{aligned}\right\} \tag{22}$$

If T_{ab} represents the material content of the spacetime and $\vec{v}$ is the four-velocity of some observer, then $\tilde{\rho}$ is the energy density as measured by such an observer, P is called the *isotropic* pressure (measured by that observer), and $\mathcal{F}^a$ and Π_{ab} are, respectively, the *momentum flux* and the *anisotropic pressure tensor* that the observer $\vec{v}$ measures. Notice that

$$\mathcal{F}^a v_a \;=\; g^{ab}\Pi_{ab} \;=\; \Pi_{ab}v^b \;=\; 0. \qquad (23)$$

Recalling now (19), one has that in the case of B_T warped spacetimes and working in an adapted (but otherwise arbitrary) chart, the Einstein tensor has got this box diagonal form. A direct inspection of the functional dependence of the components of G_{ab} above shows that given any adapted tetrad $\vec{v}$, $\vec{p}$, $\vec{y}$, $\vec{z}$ to (M,g), the Einstein, or equivalently, the energy-momentum tensor T_{ab}, may be written as

$$\begin{aligned} G_{ab} = T_{ab} = \rho v_a v_b + \mathcal{F}(v_a p_b + p_a v_b) \\ + P_1 p_a p_b + P_2(y_a y_b + z_a z_b), \end{aligned} \qquad (24)$$

for some functions

$$\left.\begin{aligned} &\rho = \omega^2\left(\tfrac{1}{2}R_2(x^\gamma) + S_1(x^D)\right), \\ &P_1 = \omega^2\left(-\tfrac{1}{2}R_2(x^\gamma) + S_3(x^D)\right), \\ &\mathcal{F} = \mathcal{F}(x^D) \quad \text{and} \quad P_2 = P_2(x^D) \end{aligned}\right\} \qquad (25)$$

Moreover, if one defines the null vector $k_a = v_a + p_a$, the above expression can be rewritten as

$$\begin{aligned} G_{ab} = T_{ab} = \mathcal{F}k_a k_b + (\rho - \mathcal{F})v_a v_b + \\ + (P_1 - \mathcal{F})p_a p_b + P_2(y_a y_b + z_a z_b). \end{aligned} \qquad (26)$$

Physically, this can be interpreted by saying that the material content of one such spacetime can always be represented either as an anisotropic fluid with four-velocity $\vec{v}$ (comoving with an adapted observer), density ρ, pressures P_1 and P_2, and momentum flow $\mathcal{F}p_a$ (equation (24)); or else (equation (26)) as the sum of an anisotropic fluid with the same four-velocity $\vec{v}$, density $\rho - \mathcal{F}$, pressures $P_\perp = P_1 - \mathcal{F}$ and P_2, plus a null radiation field directed along $\vec{k}$ carrying an energy density $\mathcal{F}$. This splitting of the energy momentum-tensor (especially the last one (26)) has been extensively used in the spherically symmetric context: see [9] and references cited therein.

It is also interesting to note that the above decompositions are highly non-unique in the sense that G_{ab} or T_{ab} can be split in a similar manner for all observers $v^{'a}$ whose world lines are tangent to M_1 everywhere (be they adapted, i.e.: $\vec{v}$ hypersurface orthogonal, or not), that is; whose four velocity is $v^{'a} = \cosh\phi v^a + \sinh\phi p^a$ for an arbitrary function $\phi(x^D, x^\gamma)$, then $p^{'a} = \sinh\phi v^a + \cosh\phi p^a$ and also $k^{'a} = v^{'a} + p^{'a}$. If ϕ depends on x^γ (i.e.: the observer $\vec{v}'$ is non-adapted) the corresponding density ρ', pressures P_1', P_2', etc. will not have the functional form (25), but if $\phi = \phi(x^D)$ alone the resulting observer and tetrad are also adapted and then (25) holds for the primed quantities ρ', etc.

B. The anisotropic pressure tensor and the shear tensor

Writing G_{ab} in equations (24, 26) in the form of equation (21) and using the adapted observer $\vec{v}$ to perform the decomposition, one has:

$$T_{ab} = \tilde{\rho} v_a v_b + P h_{ab} + \Pi_{ab} + v_a \mathcal{F}_b + \mathcal{F}_a v_b \qquad (27)$$

where

$$\left.\begin{aligned} &\tilde{\rho} = \rho, \qquad P = \tfrac{1}{3}(P_1 + 2P_2), \\ &\mathcal{F}_a = \mathcal{F}p_a, \qquad h_{ab} = p_a p_b + y_a y_b + z_a z_b, \\ &\Pi_{ab} = \tfrac{1}{3}(P_1 - P_2)\,(2p_a p_b - y_a y_b - z_a z_b) \\ &\text{or}\;\; \Pi_{ab} \equiv \Pi\left(p_a p_b - \tfrac{1}{3}h_{ab}\right) \\ &\text{with}\;\; \Pi = P_1 - P_2. \end{aligned}\right\} \qquad (28)$$

¿From (13) and the expression of Π_{ab} given above, it is now immediate to see that the shear tensor σ_{ab} of $\vec{v}$ is proportional to the anisotropic pressure tensor Π_{ab}, whenever both tensors are non vanishing:

$$\Pi_{ab} = \lambda\sigma_{ab}, \quad \text{with} \quad \lambda = \Pi^{-1}\omega\vartheta. \qquad (29)$$

If $\lambda < 0$, it can be interpreted as a shear viscosity coefficient: $\lambda = -2\eta$, $\eta > 0$ being the so called *kinematic viscosity coefficient*, and then viscosity can be seen as the source of anisotropy in the pressure.

For any other adapted observer $\vec{v}'$, with

$$v^{'a} = \cosh\phi\, v^a + \sinh\phi\, p^a, \quad p^{'a} = \sinh\phi\, v^a + \cosh\phi\, p^a$$

where $\phi = \phi(x^D)$ one obtains expressions similar to those above:

$$\tilde{\rho}' = \rho\cosh^2\phi - 2\mathcal{F}\sinh\phi\cosh\phi + P_1\sinh^2\phi,$$

$$P' = \frac{1}{3}\left(P_1' + 2P_2'\right),$$

$$P_1' = \rho\sinh^2\phi - 2\mathcal{F}\sinh\phi\cosh\phi + P_1\cosh^2\phi,$$

$$P_2' = P_2, \quad \mathcal{F}_a' = \mathcal{F}'p_a',$$

$$\mathcal{F}' = \mathcal{F}\cosh 2\phi - \frac{1}{2}(\rho + P_1)\sinh 2\phi,$$

$$h_{ab}' = p_a' p_b' + y_a y_b + z_a z_b,$$

$$\Pi_{ab} = \Pi'\left(p_a' p_b' - \frac{1}{3}h_{ab}\right), \quad \Pi' = P_1' - P_2',$$

where the primed magnitudes are those measured by $\vec{v}'$. Notice that one also has $\Pi_{ab}' = \lambda'\sigma_{ab}'$; thus, for all the adapted observers the anisotropic pressure tensor is proportional to their shear tensor. This proportionality can be tracked back to the decomposable spacetime $(M, \hat{g})$; to this end consider the adapted tetrad and adapted observer in $(M, \hat{g})$ which are conformally related to those in (M, g); i.e.: $\hat{\vec{v}} = \omega^{-1}\vec{v}, \ldots, \hat{\vec{z}} = \omega^{-1}\vec{z}$ (see previous section); from (17) we get

$$\left.\begin{aligned} \hat{G}_{AB} &= -\tfrac{1}{2}R_2(-\hat{v}_A\hat{v}_B + \hat{p}_A\hat{p}_B), \\ \hat{G}_{A\beta} &= 0 \\ \hat{G}_{\alpha\beta} &= -\tfrac{1}{2}R_1(\hat{y}_\alpha\hat{y}_\beta + \hat{z}_\alpha\hat{z}_\beta) \end{aligned}\right\} \quad (30)$$

which may also be decomposed with respect to the observer $\hat{\vec{v}}$ as in (21) thus getting

$$\hat{G}_{ab} = \hat{T}_{ab} = \hat{\rho}\hat{v}_a\hat{v}_b + \hat{P}\hat{h}_{ab} + \hat{\Pi}_{ab} + \hat{\mathcal{F}}_a\hat{v}_b + \hat{v}_a\hat{\mathcal{F}}_b, \quad (31)$$

with

$$\left.\begin{aligned} \hat{\rho} = \tfrac{1}{2}R_2, \quad \hat{P} = \tfrac{1}{3}\left(-\tfrac{1}{2}R_2 + R_1\right) \\ \hat{\mathcal{F}}_a = 0, \quad \hat{\Pi}_{ab} = \hat{\Pi}\left(\hat{p}_A\hat{p}_B - \tfrac{1}{3}\hat{h}_{ab}\right) \end{aligned}\right\} \quad (32)$$

where $\hat{\Pi} = \frac{1}{2}(R_1 - R_2)$. From the above expression for $\hat{\Pi}_{ab}$ and (11) one has $\hat{\Pi}_{ab} = \hat{\lambda}\hat{\sigma}_{ab}$, and recalling that $\sigma_{ab} = \omega\hat{\sigma}_{ab}$ and $\Pi_{ab} = \lambda\sigma_{ab}$, one finally concludes

$$\Pi_{ab} \propto \hat{\Pi}_{ab}. \quad (33)$$

The true equation of state that describes the properties of matter at densities higher than nuclear ($\approx 10^{14}$ gr/cm^3) is essentially unknown due to our inability to verify the microphysics of nuclear matter at such high densities [14]. Having this uncertainty in mind, it seems reasonable to explore some possible equations of state for the local anisotropy starting from a simple geometrical object as is the shear tensor σ_{ab}. The proportionality of the anisotropic and the shear tensors opens the possibility to devise such equations of state .

Needless to say, a decomposable spacetime of these characteristics does not represent itself any reasonable physical content (notice that $\hat{\rho} + \hat{P}_1 = 0$), however, it is still interesting to realize how this decomposable structure somehow 'generates' anisotropy in the pressures in the physically realistic warped spacetime. This is in contrast with the warping factor ω, that contributes to what one could roughly call the 'isotropic physics', namely: the energy density ρ and the isotropic pressure P.

C. Eigenvector structure and energy conditions.

Let us next see how the assumed geometry (warped spacetime) imposes certain restrictions on the material content, and how this shows up in the algebraic (eigenvector/eigenvalue) structure of the Einstein tensor.

We begin by noting that the eigenvectors of the Einstein tensor G_{ab} are the same as those of the Ricci tensor R_{ab}, their corresponding eigenvalues being 'shifted' by an amount $-\frac{1}{2}R$, where R is the Ricci scalar associated with g, furthermore, on account of the form of $\hat{G}_{ab}$ (see the remarks at the end of the preceding section) and equation (15), it follows that these eigenvectors coincide with those of the tensor $\omega_{a/b}$.

Thus, the three tensors G_{ab}, R_{ab} and $\omega_{a/b}$ all have the same Segre type [15] with the same eigenvectors. For convenience we shall work with the Ricci tensor in an adapted coordinate chart, thus we have

$$R^a_{\ b} = \begin{pmatrix} R^A_{\ B} & 0 \\ 0 & R^\alpha_{\ \beta} \end{pmatrix}, \quad (34)$$

with

$$R^A_{\ B} = R^A_{\ B}(x^D) \quad \text{and} \quad R^\alpha_{\ \beta} = f(x^D, x^\gamma)\delta^\alpha_{\ \beta}.$$

The characteristic polynomial of $R^a_{\ b}$ is then

$$p(x) = \det\left[R^a_{\ b} - x\delta^a_{\ b}\right] \quad \Rightarrow$$

$$p(x) = \det\left(R^A_{\ B}(x^D) - x\delta^A_{\ B}\right)(x - f(x^D, x^\gamma))^2 \quad (35)$$

and therefore there is one repeated eigenvalue $x = f$ that corresponds to two spacelike eigenvectors tangent to M_2 that can be chosen unit and mutually orthogonal, say $\vec{y}$ and $\vec{z}$; one therefore has in an adapted chart: $y^a = (0, 0, y^2, y^3)$ and $z^a = (0, 0, z^2, z^3)$ (furthermore: in a chart in which h_2 takes diagonal form $y^a = (0, 0, y^2, 0)$ and $z^a = (0, 0, 0, z^3)$). The remaining eigenvalues are the roots of the second degree polynomial

$$q(x) = \det\left(R^A_{\ B}(x^D) - x\delta^A_{\ B}\right) = x^2 - t_R x + d_R, \quad (36)$$

where $t_R = R^0_{\ 0} + R^1_{\ 1}$ is the trace of the matrix $(R^A_{\ B})$ and d_R is its determinant. Some elementary algebra considerations lead to the following three possibilities:

a. The polynomial $q(x)$ has two real roots. If $q(x)$ has two real roots, say λ_1 and λ_2, they will be functions on M_1 (i.e.: functions of the coordinates x^D) since $R^A_{\ B}$ are also functions on M_1. The necessary and sufficient condition for this to happen is that

$$t_R^2 - 4d_R > 0, \quad (37)$$

or, on account of our previous considerations on eigenvector/eigenvalue structure of $R^A_{\ B}$ and $\omega^A_{\ /B}$, that

$$t_\omega^2 - 4d_\omega > 0, \quad (38)$$

with

$$t_\omega = \text{trace}\left(\omega^A_{\ /B}\right) \quad \text{and} \quad d_\omega = \det\left(\omega^A_{\ /B}\right),$$

which involves only covariant derivatives of the warping function ω taken with respect to the connection of the decomposable metric. This corresponds to R^A_B being of the diagonal Segre type $\{1,1\}$ or equivalently to the existence of two non-null, mutually orthogonal eigenvectors of R^A_B (and therefore eigenvectors of R^a_b), say $\vec{u}$ and $\vec{n}$ that may be chosen unit timelike and unit spacelike respectively, which are tangent to M_1 at every point and such that, in the basis of the tangent space to M_1 formed by $\vec{u}$ and $\vec{n}$, the Jordan form of the matrix (R^A_B) is

$$R^A_B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{39}$$

In the adapted coordinate chart under consideration, these two eigenvectors are part of an adapted tetrad (i.e.: $u^a = (u^0, u^1, 0, 0)$ and $n^a = (n^0, n^1, 0, 0)$ with $u^a = u^a(x^D)$, $n^a = n^a(x^D)$), and in particular $\vec{u}$ corresponds to an adapted observer.

¿From the conditions $-u^a u_a = n^a n_a = 1$, $u^a n_a = 0$ it is easy to see that a function $\psi(x^D)$ exists such that, for the coordinate gauge introduced in (7)

$$\left.\begin{aligned} u^a &= (A^{-1}\cosh\psi, B^{-1}\sinh\psi, 0, 0), \\ n^a &= (A^{-1}\sinh\psi, B^{-1}\cosh\psi, 0, 0). \end{aligned}\right\} \tag{40}$$

and the eigenvector equations for $\vec{u}$ and $\vec{n}$ readily imply that

$$\tanh 2\psi = \frac{-2\frac{A}{B}G^t{}_x}{G^t{}_t - G^x{}_x}. \tag{41}$$

Further, a coordinate change in M_1 exists such that $\psi = 0$ and the metric still retains its diagonal form (i.e.: it still can be written as in (7)). Such an specific coordinate gauge will be called *comoving*; at this point though, we shall not assume it yet.

¿From our previous remarks, it follows that the Ricci tensor and hence the Einstein tensor, are of the diagonal Segre type with a double spacelike eigenvalue degeneracy $\{1,1(11)\}$, that is:

$$G_{ab} = \rho u_a u_b + p_1 n_a n_b + p_2\left(y_a y_b + z_a z_b\right), \tag{42}$$

which amounts to saying that it takes a diagonal matrix form in the (pseudo-orthonormal) adapted tetrad u_a, n_a, y_a, z_a. The quantities ρ, p_1, p_2 are given by

$$\rho = \left(\frac{1}{2}R_2 + S\right)\omega^2 + 2\omega\omega_{A/B}\hat{u}^A\hat{u}^B, \tag{43}$$

$$p_1 = -\left(\frac{1}{2}R_2 + S\right)\omega^2 + 2\omega\omega_{A/B}\hat{n}^A\hat{n}^B, \tag{44}$$

$$p_2 = -\left(\frac{1}{2}R_1 + S\right)\omega^2, \tag{45}$$

where $S \equiv \omega^{-1}h^{MN}(2\omega_{M/N} - 3\omega^{-1}\omega_M\omega_N)$, and $\hat{u}_a = \omega u_a$, $\hat{n}_a = \omega n_a$. In the comoving gauge alluded to above, the coordinate components of the Einstein tensor also take a matrix diagonal form (i.e.: $G_{tx} = \omega_{t/x}$, etc.), and $\hat{u}^a = (A^{-1}, 0, 0, 0)$, $\hat{n}^a = (0, B^{-1}, 0, 0)$. Einstein's field equations imply then that the energy-momentum tensor T_{ab} takes that same form. The dominant energy condition is satisfied if and only if $\rho \geq 0$ and $-\rho \leq p_i \leq \rho$ for $i = 1, 2$.

Physically, this can be interpreted by saying that there exists one (adapted) observer that moves with four-velocity $\vec{u}$ such that measures a vanishing momentum flow, energy density ρ and pressures p_1 in the direction $\vec{n}$ (which we shall call *radial* direction/pressure), and p_2 in any other spatial direction perpendicular to $\vec{n}$ (*tangential* directions/pressures). The use of the names 'radial' and 'tangential' is justified by thinking of the situation arising in spherically symmetric spacetimes (which are particular instances of those studied here), where the direction $\vec{n}$ is perpendicular to the orbits (spheres) and can therefore be identified with the radial direction, whereas the spatial directions perpendicular to that one are necessarily tangent to the spheres, hence the name 'tangential'.

Note that perfect fluids are included into this class and they are those solutions satisfying $p_1 = p_2$. From the above expressions (44,45) it is immediate to see, on account of the functional dependence of p_1 and p_2, that a necessary condition for this to happen is that $R_2 =$ constant. Thus we have the result that *Perfect fluid type B warped spacetimes are necessarily spherically, plane or hyperbolically symmetric.*

If the matter content is described by the energy momentum tensor (28) this implies, again on account of our considerations on the eigenvector/eigenvalue structure of G^a_b, R^a_b, etc., that

$$t_G^2 - 4d_G > 0, \tag{46}$$

with

$$t_G = \operatorname{trace}\left(G^A_B\right), \quad \text{and} \quad d_G = \det\left(G^A_B\right),$$

or, in terms of the physical quantities introduced in (28):

$$\left|\frac{2\mathcal{F}}{\tilde{\rho} + P + \frac{2}{3}\Pi}\right| < 1 \tag{47}$$

These results can also be arrived at from (24) by writing

$$\left.\begin{aligned} v_a &= \cosh\phi\, u_a + \sinh\phi\, n_a \quad \text{and} \\ p_a &= \sinh\phi\, u_a + \cosh\phi\, n_a \end{aligned}\right\} \tag{48}$$

and then demanding that ϕ is such that the term in T_{ab} containing the mixed terms $u_a n_b + n_a u_b$ vanishes. This is equivalent to saying that there exists a privileged observer that measures zero momentum flow. Such an observer is moving with four-velocity

$$\left.\begin{aligned} u^a &= \cosh\phi\, v^a + \sinh\phi\, p^a \quad \text{where} \\ \tanh 2\phi &= -\frac{2\mathcal{F}}{\tilde{\rho} + P + \frac{2}{3}\Pi}. \end{aligned}\right\} \tag{49}$$

Notice that from the remarks following equation (24), it follows that $\tilde{\rho}+P+\frac{2}{3}\Pi$ is a function of the coordinates x^D, and so is $\mathcal{F}$, hence $\phi=\phi(x^D)$ which is the condition for $\vec{u}$ being an adapted observer.

The quantities $\tilde{\rho}$, P, Π and $\mathcal{F}$ in (28) and ρ, p_1 and p_2 in (42) are related through:

$$\tilde{\rho}=\rho\cosh^2\phi+p_1\sinh^2\phi,$$

$$P=\frac{1}{3}\left(\rho\sinh^2\phi+p_1\cosh^2\phi+2p_2\right),$$

$$\Pi=\rho\sinh^2\phi+p_1\cosh^2\phi-p_2,$$

$$\mathcal{F}=(\rho+p_1)\sinh\phi\cosh\phi,$$

or equivalently

$$\rho=\frac{1}{2}\left[\sqrt{\left(\tilde{\rho}+P+\frac{2}{3}\Pi\right)^2-4\mathcal{F}^2}+\tilde{\rho}-P-\frac{2}{3}\Pi\right],$$

$$p_1=\frac{1}{2}\left[\sqrt{\left(\tilde{\rho}+P+\frac{2}{3}\Pi\right)^2-4\mathcal{F}^2}-\tilde{\rho}+P+\frac{2}{3}\Pi\right],$$

$$p_2=P-\frac{1}{3}\Pi,$$

and therefore the dominant energy condition reads in these variables (recall we are assuming that (47) holds):

$$\rho\geq 0$$
$$\Downarrow$$
$$\sqrt{\left(\tilde{\rho}+P+\frac{2}{3}\Pi\right)^2-4\mathcal{F}^2}+\tilde{\rho}-\left(P+\frac{2}{3}\Pi\right)\geq 0, \quad (50)$$

$$-\rho\leq p_1\leq\rho$$
$$\Downarrow$$
$$\tilde{\rho}-\left(P+\frac{2}{3}\Pi\right)\geq 0, \quad (51)$$

$$-\rho\leq p_2\leq\rho$$
$$\Downarrow$$
$$\tilde{\rho}+P-\frac{4}{3}\Pi+\sqrt{\left(\tilde{\rho}+P+\frac{2}{3}\Pi\right)^2-4\mathcal{F}^2}\geq 0 \quad (52)$$

and

$$\tilde{\rho}+\sqrt{\left(\tilde{\rho}+P+\frac{2}{3}\Pi\right)^2-4\mathcal{F}^2}-3P\geq 0 \quad (53)$$

with

$$\left(\tilde{\rho}+P+\frac{2}{3}\Pi\right)^2-4\mathcal{F}^2\geq 0 \quad (54)$$

Notice that the second inequality (51) above implies the first one (50), therefore only the four last inequalities need be taken into account.

b. The polynomial $q(x)$ has only one real root. If $q(x)$ has just one real root, then it must be that

$$t_R^2-4d_R=0,\quad\text{or equivalently,}\quad t_\omega^2-4d_\omega=0, \quad (55)$$

where the definitions are the same as in the previous case. The Ricci (Einstein, $\omega^a_{\ b}$, etc.) tensor has then a null eigenvector $\vec{k}$ with corresponding eigenvalue (in the case of the Ricci tensor) $-\sigma=\frac{1}{2}t_R$, and the Jordan form of the matrix $(R^A_{\ B})$ is

$$R^A_{\ B}=\begin{pmatrix}-\sigma & 0\\ 1 & -\sigma\end{pmatrix}. \quad (56)$$

The whole tensor G_{ab} is then of the Segre type $\{2,(11)\}$ and therefore may be written as

$$G_{ab}=\sigma\left(k_al_b+l_ak_b\right)+\lambda k_ak_b+p_2\left(y_ay_b+z_az_b\right), \quad (57)$$

where $k_ak^a=l_al^a=0$ and $k_al^a=-1$, thus $\vec{k},\vec{l},\vec{y},\vec{z}$ form a null tetrad, and $\vec{k},\vec{l}$ may be chosen so that their components are functions on M_1 (i.e.: depend only on the coordinates x^D). The functions σ,λ and p_2 are given by:

$$\sigma=-\left(\frac{1}{2}R_2+S\right)\omega^2+2\omega\omega_{A/B}\hat{k}^A\hat{l}^B, \quad (58)$$

$$\lambda=2\omega\omega_{A/B}\hat{l}^A\hat{l}^B,\ \ p_2=-\left(\frac{1}{2}R_1+S\right)\omega^2, \quad (59)$$

where, as in the previous case $S\equiv\omega^{-1}h^{MN}(2\omega_{M/N}-3\omega^{-1}\omega_M\omega_N)$, and $\hat{k}_a=\omega k_a$, $\hat{l}_a=\omega l_a$. In this case one has that $\omega_{A/B}\hat{k}^A\hat{k}^B=0$. It is easy to see that coordinates $\{u,v,y,z\}$ exist such that the decomposable metric can be written as

$$d\hat{s}^2=-2B^2(u,v)dudv+e^{2Q(y,z)}\left(dy^2+dz^2\right), \quad (60)$$

and then $\hat{k}^a=(B^{-1},0,0,0)$, $\hat{l}^a=(0,B^{-1},0,0)$. In this coordinate gauge, the equation $\omega_{A/B}\hat{k}^A\hat{k}^B=0$ reads simply $\omega_{u/u}=0$, which can be easily integrated once to get $\omega_u=B^2$, where a redefinition of the coordinate v has been carried out in order to dispose of one non-essential function of v that appears when integrating the previous equation.

Any pair $\vec{u}$, $\vec{n}$ of mutually orthogonal, unit (timelike and spacelike respectively) vector fields contained in the two-space spanned by $\vec{k}$ and $\vec{l}$ will be of the form

$$u_a=\frac{a}{\sqrt{2}}\left(k_a+\frac{1}{a^2}l_a\right)\quad\text{and}\quad n_a=\frac{a}{\sqrt{2}}\left(k_a-\frac{1}{a^2}l_a\right), \quad (61)$$

where a is some arbitrary function; it turns then out that G_{ab} above can be rewritten, in terms of the pseudo-orthonormal tetrad $\vec{u}, \vec{n}, \vec{y}, \vec{z}$ as

$$G_{ab} = \left(\sigma + \frac{\lambda}{2a^2}\right) u_a u_b + \left(\frac{\lambda}{2a^2} - \sigma\right) n_a n_b + \frac{\lambda}{2a^2}\left(u_a n_b + n_a u_b\right) + p_2\left(y_a y_b + z_a z_b\right), \quad (62)$$

and the dominant energy condition is satisfied if and only if [15]

$$\sigma \geq 0, \quad \lambda > 0 \quad \text{and} \quad -\sigma \leq p_2 \leq \sigma. \quad (63)$$

As in the previous case, if the matter content is described by the energy momentum tensor (24) this implies that

$$\sigma + \frac{\lambda}{2a^2} = \tilde{\rho}, \quad \frac{\lambda}{2a^2} - \sigma = P + \frac{2}{3}\Pi, \quad \frac{\lambda}{2a^2} = \mathcal{F}, \quad (64)$$

which readily implies

$$\left|\frac{2\mathcal{F}}{\tilde{\rho} + P + \frac{2}{3}\Pi}\right| = 1 \quad \Leftrightarrow \quad \left|\tilde{\rho} + P + \frac{2}{3}\Pi\right| = |2\mathcal{F}|. \quad (65)$$

In this case, no coordinate system exists such that G_{ab} takes a diagonal form; or, put into physical language, all allowed physical observers will always measure a non-vanishing momentum flow $\mathcal{F}$, but $\mathcal{F}$ must satisfy equation (65); further, the dominant energy condition (63) can be translated as

$$\tilde{\rho} - \left(P + \frac{2}{3}\Pi\right) \geq 0 \quad \text{and} \quad \mathcal{F} > 0. \quad (66)$$

Again in this case, we note the proportionality between Π_{ab} and σ_{ab}.

c. The polynomial $q(x)$ has two complex roots. If $q(x)$ admits two complex roots they must necessarily complex conjugate of one another, say z and $\bar{z}$. In this case it is well known [15] that the dominant energy condition cannot be satisfied, consequently, if T_{ab} is of this type it cannot represent physically acceptable matter. We shall not consider this case any further, but note in passing that this would arise whenever

$$\left|\frac{2\mathcal{F}}{\tilde{\rho} + P + \frac{2}{3}\Pi}\right| > 1. \quad (67)$$

D. Summarizing some of the results

In order to close this section, we summarize the results thus far obtained as follows:

1. The only possible cases compatible with a type B_T warped geometry which satisfy the dominant energy condition correspond to G_{ab} (or T_{ab}) being of the type $\{1, 1(11)\}$ or $\{2, (11)\}$ (or any degeneracy thereof). In both cases, the material content of the spacetime can be interpreted (by any adapted observer) either as an anisotropic fluid with momentum flow, or else as the sum of an anisotropic fluid with no momentum flow and a pure radiation field.

2. It is of the type $\{1, 1(11)\}$ whenever (47), and then the inequalities (51) through (54) must be satisfied in order to fulfill the dominant energy condition. In any case, and for any adapted observer (including the privileged one that sees no momentum flux), proportionality exists between the anisotropic pressure and shear tensors of these observers. Perfect fluid spacetimes are of the type $\{1, (111)\}$ and one then has R_2 = constant; i.e.: the spacetime is spherically, plane or hyperbolically symmetric.

3. It is of the type $\{2, (11)\}$ whenever (65) holds, then (66) must hold in order to satisfy the dominant energy condition. Again, proportionality exists between the anisotropic pressure and shear tensors of adapted observers.

IV. RADIATION HYDRODYNAMICS SCENARIO.

In this section we are going to present some of the consequences of the general results on the energy conditions and the structure of the energy momentum tensor obtained above. We shall particularize to the case of spherical symmetry (which, as previously discussed is a particular case of warped B_T spacetime) and consider a radiating fluid, but from our previous discussion it should become clear that all the results obtained in this section are immediately generalizable to the case of a generic warped B_T spacetime. In this case the energy-momentum tensor could describe

- An anisotropic (non-pascalian) fluid of velocity $\vec{v}$ (assumed rotation-invariant and therefore adapted in the sense defined previously) and energy-momentum tensor $T^{M\ (b)}_{(a)} = diag\,(\rho, P_r, P_\perp, P_\perp)$, where ρ is the energy density, P_r the radial pressure and $P_\perp$ the tangential pressure. The indices enclosed within round brackets are tetrad indices, the tetrad being $\vec{v}, \vec{p}, \vec{y}, \vec{z}$, where $\vec{y}, \vec{z}$ are mutually orthogonal, unit, spacelike and tangent to the spherical orbits, $\vec{p}$ is unit spacelike and perpendicular to the spheres, and $\vec{v}$ is unit timelike and orthogonal to the previous three.

- A radiation field of specific intensity $\mathbf{I}(x, t; \vec{n}, \nu)$ given through

$$\mathrm{d}\mathcal{E} = \mathbf{I}(r, t; \vec{n}, \nu)\mathrm{d}S\ \cos\varphi\ \mathrm{d}\Theta\ \mathrm{d}\nu\ \mathrm{d}t, \quad (68)$$

where $\mathrm{d}\mathcal{E}$ is defined as the energy crossing a surface element $\mathrm{d}S$, into the solid angle around $\vec{n}$, i.e.

$\mathrm{d}\Theta \equiv \sin\theta \mathrm{d}\theta \mathrm{d}\psi \equiv -\mathrm{d}\mu \mathrm{d}\psi$ (φ is the angle between $\vec{n}$ and the normal to $\mathrm{d}S$), transported by a radiation of frequencies $(\nu, \nu + \mathrm{d}\nu)$ in time $\mathrm{d}t$. It is measured at the position x and time t, traveling in the direction $\vec{n}$ with a frequency ν. As in classical radiative transfer theory, for a planar geometry, the moments of $\mathbf{I}(x, t; \vec{n}, \nu)$ can be written as [6, 16, 17]

$$\rho_R = \frac{1}{2}\int_0^\infty d\nu \quad \int_1^{-1} d\mu \;\; \mathbf{I}(x, t; \vec{n}, \nu), \tag{69}$$

$$\mathcal{F} = \frac{1}{2}\int_0^\infty d\nu \quad \int_1^{-1} d\mu \;\; \mu\, \mathbf{I}(x, t; \vec{n}, \nu) \tag{70}$$

and

$$\mathcal{P} = \frac{1}{2}\int_0^\infty d\nu \quad \int_1^{-1} d\mu \;\; \mu^2 \mathbf{I}(x, t; \vec{n}, \nu) \quad . \tag{71}$$

Physically, ρ_R , $\mathcal{F}$ and $\mathcal{P}$, represent the radiation contribution to the energy density, energy flux density and radial pressure, respectively.

¿From the above assumptions the energy-momentum tensor can be written as $T_{(a)(b)} = T^M_{(a)(b)} + T^R_{(a)(b)}$ where the material part is $T^M_{(a)(b)}$ given above, and the corresponding term for the radiation field $T^R_{(a)(b)}$ can be written, in the tetrad introduced, as [6, 16, 17]

$$T^R_{(a)(b)} = \begin{pmatrix} \rho_R & \mathcal{F} & 0 & 0 \\ \mathcal{F} & \mathcal{P} & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\rho_R - \mathcal{P}) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(\rho_R - \mathcal{P}) \end{pmatrix} . \tag{72}$$

therefore in this case the generic physical variables are

$$\left.\begin{aligned} \tilde{\rho} &= \rho + \rho_R; \\ P &= \tfrac{1}{3}\left(P_r + 2P_\perp + \rho_R\right) \\ \Pi &= P_r + \tfrac{3}{2}\mathcal{P} - P_\perp - \tfrac{1}{2}\rho_R \end{aligned}\right\} \tag{73}$$

In coordinates, the energy-momentum tensor can be written as:

$$\begin{aligned} T_{ab} = &(\rho + \rho_R) v_a v_b + (P_r + \mathcal{P}) p_a p_b + \\ &+ \mathcal{F}(v_a p_b + p_a v_b) \\ &+ \frac{1}{2}(P_\perp + \rho_R - \mathcal{P})(y_a y_b + z_a z_b) \end{aligned} \tag{74}$$

or, using the notation set up in (21):

$$\begin{aligned} T_{ab} = &(\rho + \rho_R) v_a v_b + \frac{1}{3}(P_r + 2P_\perp + \rho_R) h_{ab} \\ &+ \mathcal{F}(v_a p_b + p_a v_b) \\ &+ \left(P_r - P_\perp + \frac{1}{2}(3\mathcal{P} - \rho_R)\right)(p_a p_b - \frac{1}{3}h_{ab}), \end{aligned} \tag{75}$$

where the last term shall be often written in the calculations as

$$\Pi_{ab} = \left(P_r - P_\perp + \frac{1}{2}(3\mathcal{P} - \rho_R)\right)(p_a p_b - \frac{1}{3}h_{ab})$$
$$\Pi_{ab} \equiv \Pi(p_a p_b - \frac{1}{3}h_{ab}),$$

and the notation established in the previous section will also be used:

$$\tilde{\rho} = \rho + \rho_R, \quad P = \frac{1}{3}\left(P_r + 2P_\perp + \rho_R\right),$$
$$\Pi = \left(P_r - P_\perp + \frac{1}{2}\left(3\mathcal{P} - \rho_R\right)\right).$$

>¿From a physical point of view, the above tensor represents the most general situation one is interested in an astrophysical scenario, and will therefore be adopted as representing the matter content from now on. Also note that, from our developments in the preceding sections, it follows that the spacetime geometry “forces” this kind of matter content (note that not all of the possible combinations of energy-momentum tensors give rise to a total energy-momentum tensor compatible with the warped geometry; i.e.: of the types $\{1, 1(11)\}$ or $\{2, (11)\}$, see [15] for further details).

Next we shall translate to the present case the conditions we obtained in general; i.e.: (47) and (65), together with the corresponding inequalities (51) through (54) and (66) for the dominant energy condition.

For the case of G_{ab} (or T_{ab}) being of the type $\{1, 1(11)\}$ (47) can be rewritten as

$$\left|\frac{2\mathcal{F}}{\rho + \rho_R + P_r + \mathcal{P}}\right| < 1 \tag{76}$$

and the inequalities (51) through (54) yield

$$\rho + \rho_R - P_r - \mathcal{P} \geq 0, \tag{77}$$

$$\begin{aligned} &\rho + 2\rho_R - P_r + 2P_\perp - 2\mathcal{P} + \\ &+ \sqrt{(\rho + \rho_R + P_r + \mathcal{P})^2 - 4\mathcal{F}^2} \geq 0 \end{aligned} \tag{78}$$

$$\rho - P_r - 2P_\perp + \sqrt{(\rho + \rho_R + P_r + \mathcal{P})^2 - 4\mathcal{F}^2} \geq 0 \tag{79}$$

$$(\rho + \rho_R + P_r + \mathcal{P} - 2\mathcal{F})(\rho + \rho_R + P_r + \mathcal{P} + 2\mathcal{F}) \geq 0 \tag{80}$$

or

$$\left|\frac{2\mathcal{F}}{\bar{\rho} + \bar{P}_r}\right| < 1 \tag{81}$$

and

$$\left.\begin{aligned}\sqrt{\left(\bar{\rho}+\bar{P}_r\right)^2-4\mathcal{F}^2}+\bar{\rho}-\bar{P}_r\geq 0,\\ \bar{\rho}-\bar{P}_r\geq 0,\\ \bar{\rho}-\bar{P}_r+2\bar{P}_\perp+\sqrt{\left(\bar{\rho}+\bar{P}_r\right)^2-4\mathcal{F}^2}\geq 0\\ \bar{\rho}-\bar{P}_r-2\bar{P}_\perp+\sqrt{\left(\bar{\rho}+\bar{P}_r\right)^2-4\mathcal{F}^2}\geq 0\\ \left(\bar{\rho}+\bar{P}_r+2\mathcal{F}\right)\left(\bar{\rho}+\bar{P}_r-2\mathcal{F}\right)\geq 0\end{aligned}\right\} \quad (82)$$

where we defined

$$\bar{\rho}=\rho+\rho_R,\quad \bar{P}_r=P_r+\mathcal{P},\quad \text{and}\quad \bar{P}_\perp=P_\perp+\frac{1}{2}(\rho_R-\mathcal{P})$$

which represent the "total" density, radial pressure and tangential pressure as measured by a local minkowskian observer.

Concerning the case in which G_{ab} (or T_{ab}) is of the type $\{2,(11)\}$, (65) can be rewritten as

$$|\rho+\rho_R+P_r+\mathcal{P}|=|2\mathcal{F}| \quad (83)$$

and the inequalities (66)

$$\rho+\rho_R-P_r-\mathcal{P}\geq 0 \quad \text{and} \quad \mathcal{F}>0. \quad (84)$$

or equivalently

$$\left|\bar{\rho}+\bar{P}_r\right|=|2\mathcal{F}| \quad \text{and} \quad \left\{\begin{aligned}\bar{\rho}-\bar{P}_r\geq 0\\ \mathcal{F}>0\end{aligned}\right. .$$

V. QUASI-SPHERICAL METRICS.

In this section we shall workout two warped spacetimes which can be considered quasi-spherical metrics in the sense that we can recover the spherical line element switching of a parameter. The first of such a line element is given by:

$$ds^2=r^2\left\{-\frac{1}{2r^2}\frac{Q^2(t,r)}{P^2(t,r)}dt^2 + \frac{1}{2r^2}P^2(t,r)dr^2+\left[d\theta^2+f^2(\theta)d\phi^2\right]\right\} \quad (85)$$

where r,θ,ϕ are the usual spherical coordinates. The form of the metric coefficients g_{tt} and g_{rr} is chosen as above for convenience, but it should be clear that this can always be done without loss of generality.

It is clear that if $f(\theta)=\sin\theta$ the metric (85) describes a typical spherically symmetric spacetime[19], but if $f(\theta)\neq\sin\theta$ the above line element corresponds to a special case of axially (not spherically) symmetric spacetime. In terms of the descomposable metric structure (4) $\omega=r^{-1}$.

Let us next choose the function $f(\theta)$ as the Airy function

$$f(\theta)=\mathrm{Ai}\left(\frac{-1-a\theta}{a^{2/3}}\right)$$

where a is some arbitrary (real) parameter.

A direct calculation of $t_G^2-4d_G$ (that must be greater than or equal to 0 in order to have the types $\{1,1(11)\}$ or $\{2,(11)\}$ respectively) yields:

$$\Delta\equiv t_G^2-4d_G=16\frac{Q_r^2-4P^2P_t^2}{r^2P^4Q^2} \quad (86)$$

and

$$\Delta\geq 0 \qquad \Leftrightarrow \qquad Q_r^2-4P^2P_t^2\geq 0. \quad (87)$$

We shall assume that it is non-zero and put $\delta^2\equiv Q_r^2-4P^2P_t^2>0$; thus the Einstein tensor admits a unit timelike eigenvector (4-velocity of the preferred adapted observer that measures zero momentum flow) whose components are easily seen to be (see equations (40) and (44)):

$$u^a=\left(\frac{P}{Q}\sqrt{\frac{Q_r}{\delta}+1},\frac{1}{P}\sqrt{\frac{Q_r}{\delta}-1},0,0\right). \quad (88)$$

The spacelike unit vector $\vec{n}$ is:

$$n^a=\left(\frac{P}{Q}\sqrt{\frac{Q_r}{\delta}-1},\frac{1}{P}\sqrt{\frac{Q_r}{\delta}+1},0,0\right). \quad (89)$$

The density ρ and pressures p_1 and p_2 measured by $\vec{u}$ are now:

$$\rho=\frac{1}{r^2QP^3\delta}\left\{\delta\left[2r(2P_rQ-PQ_r)+QP^3-2QP\right] + 2rP(4P^2P_t^2+Q_r^2)\right\}+\frac{a}{r^2}\theta \quad (90)$$

$$p_1=\frac{-1}{r^2QP^3\delta}\left\{\delta\left[2r(2P_rQ-PQ_r)+QP^3-2QP\right] - 2rP(4P^2P_t^2+Q_r^2)\right\}-\frac{a}{r^2}\theta \quad (91)$$

$$p_2=\frac{2}{rQ^3P^4}\left\{Q^2P(PQ_r-2QP_r)+3rQ^2P_r(QP_r-Q_rP) + 3rQ^2P_r(QP_r-Q_rP)+rQ^2P(PQ_{rr}-QP_{rr}) -rP^4(PQP_{tt}-PP_tQ_t+P_t^2Q)\right\} \quad (92)$$

The second line element can has the form of

$$ds^2=r^2\left\{-\frac{1}{2r^2}\frac{Q^2(t,r)}{P^2(r)}dt^2 + \frac{1}{2r^2}P^2(r)dr^2+\left[f^2(\theta)d\theta^2+\sin^2(\theta)\,d\phi^2\right]\right\} \quad (93)$$

again r, θ, ϕ are the usual spherical coordinates. and if $f(\theta) = 1$ the metric (93) describes a spherically symmetric spacetime.

Now choosing the function $f(\theta)$ as

$$f^2(\theta) = \frac{2\cos^2(\theta)}{2\cos^2\theta - ((1 - 2\cos^2\theta)\,\theta + \cos\theta\sin\theta)\,a}$$

we get

$$\Delta \equiv t_G^2 - 4d_G = 16\,\frac{Q_r^2}{P^4 r^2 Q^2} \tag{94}$$

The 4-velocity of the preferred adapted observer and the spacelike unit vector $\vec{n}$ now are easily seen to be

$$u^a = \left(\frac{\sqrt{2}P}{Q}, 0, 0, 0\right) \quad \text{and} \quad n^a = \left(0, \frac{\sqrt{2}}{P}, 0, 0\right), \tag{95}$$

respectively

The density ρ and pressures p_1 and p_2 measured by $\vec{u}$ are now written as:

$$\rho = \frac{4P_r}{P^3 r} + \frac{1}{r^2} - \frac{2}{P^2 r^2} + \frac{a\theta}{r^2}, \tag{96}$$

$$p_1 = \frac{4\,Q_r}{P^2 r Q} - \frac{4P_r}{P^3 r} - \frac{1}{r^2} + \frac{2}{P^2 r^2} - \frac{a\theta}{r^2} \tag{97}$$

and

$$p_2 = \frac{2Q_{rr}}{P^2 Q} - \frac{2P_{rr}}{P^3} - \frac{4P_r}{P^3 r} + \frac{2Q_r}{P^2 r Q} + \frac{6{P_r}^2}{P^4} - \frac{6Q_r P_r}{P^3 Q} \tag{98}$$

It is interesting to see that in both cases the values of ρ and p_1 for $a = 0$, correspond to the spherical case. Thus the terms $ar^{-2}\theta$ can be seen as the contribution of the "lack of sphericity" which comes from the decomposable part of each metric modifying the density and the radial pressure but having no effect on the tangential pressure p_2.

All the energy conditions (50) through (54) can be satisfied and in both cases because

$$\rho > 0 \quad \Rightarrow \begin{cases} \text{if } a \leq 0 & \Rightarrow \rho_{Spheric} \geq \rho_{QSpheric} \\ \text{if } a \geq 0 & \Rightarrow \rho_{Spheric} \leq \rho_{QSpheric} \end{cases}$$

with $\rho_{Spheric}$ corresponding to the density with $a = 0$ measured at any point and $\rho_{QSpheric}$ the density for the quasispheric case.

It should be stressed that both are exact solutions having the parameter a of any order, in particular if $a \sim 0$, it could be considered as a perturbation of the spherical solution

VI. CONCLUSIONS

We have studied in detail class B_T warped spacetimes which include as a special case all the spherically symmetric solutions of the Einstein's Field equations.

We have shown that the Segre type of the Einstein and energy-momentum tensor of such spacetimes can only be $\{1, 1(11)\}$, $\{2, (11)\}$ or $\{z\bar{z}, (11)\}$ (or any degeneracy of these types), the latter being non-admissible on physical grounds (the dominant energy condition is violated).

We have given algebraic conditions for these types involving only covariant derivatives of the warping function ω with respect to the underlying decomposable metric $\hat{g}$; namely $t_\omega^2 - 4d_\omega$ greater than, equal to or less than 0 respectively, see equations (38,55) (alternatively, they can also be characterized in terms of the components of the Einstein or the Ricci tensors: $t_G^2 - 4d_G \geq 0$, etc.), and we have provided expressions for the eigenvectors of the Einstein tensor in the two physically relevant cases: $\{1, 1(11)\}$ and $\{2, (11)\}$. Further, we have provided explicit algebraic expressions for the inequalities stemming from the Dominant Energy Condition in each of the above two cases. It has also been shown that if the matter content is to be a perfect fluid (i.e.: $\{1, (111)\}$), the spacetime must then necessarily be spherically, hyperbolically or plane symmetric.

We have introduced the concepts of *adapted tetrads* and *adapted observers*, and shown that the eigenvectors of the Einstein tensor alluded to above always form an adapted tetrad, hence the timelike eigenvector in the case $\{1, 1(11)\}$ corresponds to an adapted observer.

The preceding mathematical developments, have been linked to physics by showing that the material content of such spacetimes can always be interpreted by *all* of the adapted observers either as an anisotropic fluid with momentum flow, or as the sum of an anisotropic fluid with zero momentum flow plus a radiation field. Moreover, we have shown that the anisotropic pressure tensor is always proportional to the shear of the observer who measures that anisotropy, this suggesting a model for an equation of state. Adapted observers appear then as the most natural observers in these spacetimes. We also translated both the conditions leading to either Segre type of the Einstein tensor and the restrictions imposed by the Dominant Energy Condition, in terms of the various physical magnitudes measured by any adapted observer; see (47, 50 to 53) and (65, 66). It can be noted in passing that the underlying decomposable spacetime structure is somehow related to the anisotropy, whereas the warping factor is related to what one could call the 'isotropic physics'.

The radiation hydrodynamics scenario has been discussed in some detail, showing that these spacetimes can accommodate quite naturally all of the physical components that a material content described by a radiating fluid has, meeting all the requirements that such an scenario demands. Again, we expressed the various conditions in terms of the physical variables.

Finally, we presented two examples of non-spherically symmetric metrics which depend upon an arbitrary parameter a, such that for $a = 0$ spherical symmetry is recovered in both cases. Interestingly enough, the expressions for various physical quantities (density, pressures, ...) split up nicely in a part that does not contain a (i.e.: the values one would obtain if the spacetime were spherically symmetric) plus a term proportional to a, accounting for the 'lack of sphericity'.

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[1] J. Carot and J. da Costa, *Class. Quantum Grav.,* 10, 461, (1993).

[2] G. S. Hall and W. Kay, *J. Math. Phys*., 29, 420 (1988).

[3] J. Carot and B.O. J. Tupper *Class. Quantum Grav.* 19, 4141, (2002).

[4] H. Stephani, D. Kramer, M.A.H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions to Einstein's Field Equations* 2nd Edition, Cambridge University Press, Cambridge, (2004).

[5] L. Herrera, and N.O. Santos, *Physics Reports,* 286, 53, (1997).

[6] Mihalas, D. and Weibel Mihalas, B. (1984) *Foundations of Radiation Hydrodynamics* (Oxford University Press).

[7] Mihalas, D. (2004) *A Bibliography of Literature on Radiation Hydrodynamics*, IAU Commission 36 Theory of Stellar Atmospheres, at `http://dasgal.obspm.fr/IAU36/radhydro.pdf`

[8] M. Ishak and K. Lake, *Phys Rev D***68**, 104031 (2003) also http://arxiv.org/abs/gr-qc/0304065.

[9] L. Herrera, A. Di Prisco, J. Martin, J. Ospino, N.O. Santos, and O. Troconis, *Phys Rev D* **69**, 084026 (2004), also http://arxiv.org/abs/gr-qc/0403006.

[10] B. O'Neill, *Semi-Riemannian Geometry,* New York: Academic, (1983).

[11] J.K. Beem and P.E. Erhlich, *Global Lorentzian Geometry*, New York: Marcel Dekker, (1981).

[12] B. Haddow and J. Carot, *Class. Quantum Grav.* **13**, 289, (1996).

[13] R.M. Wald, *General Relativity*, The University of Chicago Press, Chicago, (1984).

[14] N. K. Glendenning, , *Compact Stars,* New York, Springer Verlag, (2000),

[15] G.S. Hall and D.A. Negm, *Int. J. of Theor. Phys.* **25**, 405 (1986).

[16] Lindquist R. W. (1966) *Ann. Phys.* **37,** 487.

[17] Rezzolla L. and Miller J. (1994) *Class. Quantum Grav.* **11,** 1815.

[18] Frank W. J. Olver (2001) *AI Airy and Related Functions* in Digital Library of Mathematical Functions `http://dlmf.nist.gov/Contents/AI/AI.9_iv.html`

[19] In the region where the normal to the spherical orbits is spacelike.