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Nonlocal Equation of State in General Relativistic Radiating Spheres

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We show that under particular circumstances a general relativistic spherically symmetric bounded distribution of matter could satisfy a nonlocal equation of state. This equation relates, at a given point, the components of the corresponding energy momentum tensor not only as function at that point, but as a functional throughout the enclosed configuration. We have found that these types of dynamic bounded matter configurations, with constant gravitational potentials at the surface, admit a Conformal Killing Vector and fulfill the energy conditions for anisotropic imperfect fluids. We also present several analytical and numerical models satisfying these equations of state which collapse as reasonable radiating anisotropic spheres in general relativity.

I. INTRODUCTION

Our comprehension of the behavior of highly compact stars is intimately related to the understanding of the physics at supranuclear densities. Today, the properties of matter at densities higher than nuclear ($\approx 10^{14}\ gr/cm.^3$) are essentially unknown, although they must be oriented by the experimental insight and extrapolations emerging from the ultra high energy accelerators and firmly anchored to what is known from nuclear physics [1,2]. Having this uncertainty in mind, we shall explore what is allowed by the laws of physics, for a particular equation of state, within the framework of the theory of General Relativity and considering spherical symmetry.

We shall consider a spherically symmetric space-time that can be described by the metric

$$ds^2 = h\, e^{4\beta} dT^2 - \frac{1}{h} dR^2 - R^2 d\Omega^2\,, \tag{1}$$

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where the solid angle is $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ and both β and h depend on T and R.

We have found that if in the above space-time (1) an additional restriction,

$$h(T,R) \equiv 1 - \frac{2m(T,R)}{R} = \mathrm{C}(T)\, e^{-2\beta(T,R)}, \quad \text{with} \quad 0 < \mathrm{C}(T) < 1 \ , \tag{2}$$

is considered, we can obtain the following relation between two of the components of the corresponding Einstein tensor. i.e.

$$\mathbf{G}^T_T + 3\mathbf{G}^R_R + R\frac{\partial}{\partial R}\left[\mathbf{G}^T_T + \mathbf{G}^R_R\right] = 0\,. \tag{3}$$

Equation (3) can be integrated yielding

$$\mathbf{G}^T_T = -\mathbf{G}^R_R - \frac{2}{R}\int_0^R \mathbf{G}^R_R \, \mathrm{d}\bar{r} + \frac{\mathcal{K}(T)}{R} \ , \tag{4}$$

which, using the Einstein equations, can be re-written as

$$\mathbf{T}^T_T = -\mathbf{T}^R_R - \frac{2}{R}\int_0^R \mathbf{T}^R_R \, \mathrm{d}\bar{r} + \frac{\mathcal{K}(T)}{R} \ , \tag{5}$$

where $\mathcal{K}(T)$ is an arbitrary integration function. Obviously, from (3), we can also solve $\mathbf{G}^R_R$ in order to obtain an expression equivalent to (5), i.e.

$$\mathbf{T}^R_R = -\mathbf{T}^T_T + \frac{2}{R^3}\int_0^R \bar{r}^2\mathbf{T}^T_T \, \mathrm{d}\bar{r} \ - \frac{\mathcal{G}(T)}{R^3} \ . \tag{6}$$

Again, $\mathcal{G}(T)$ is an arbitrary integration function. Clearly, for bounded matter distributions both functions $\mathcal{K}(T)$ and $\mathcal{G}(T)$, should emerge from the junction conditions to match the corresponding exterior metric.

It can be easily appreciated that the above equations (5) and (6) involve a non local relation between the components of the energy momentum tensor. At a given instant of time, one of the component (say $\mathbf{T}^T_T$ in (5)) is not only a function of the other ($\mathbf{T}^R_R$) at that point but also its functional throughout the rest of the configuration. In the case of radiating fluids, equations (5) and (6) relate non-locally the hydrodynamic and radiation physical variables.

Equation (6) could be further re-written as

$$\mathbf{T}^R_R = -\mathbf{T}^T_T + \frac{2}{3}\left\langle \mathbf{T}^T_T \right\rangle - \frac{\mathcal{G}(T)}{R^3} \ , \tag{7}$$

with

$$\left\langle \mathbf{T}^T_T \right\rangle = \frac{1}{\frac{4\pi}{3}R^3}\int_0^R 4\pi\bar{r}^2\mathbf{T}^T_T \, \mathrm{d}\bar{r} \ , \tag{8}$$

clearly the nonlocal term represents an average of the function $\mathbf{T}^T_T$ over the volume enclosed by the radius R. Moreover, equation (7) can be easily rearranged as

$$\mathbf{T}^R_R = -\frac{1}{3}\mathbf{T}^T_T - \frac{2}{3}\ \left(\mathbf{T}^T_T - \left\langle \mathbf{T}^T_T \right\rangle\right)\ - \frac{\mathcal{G}(T)}{R^3} = -\frac{1}{3}\mathbf{T}^T_T - \frac{2}{3}\,\sigma_{\mathbf{T}^T_T} - \frac{\mathcal{G}(T)}{R^3} \ , \tag{9}$$

where we have used the concept of statistical standard deviation $\sigma_{\mathbf{T}^T_T}$ from the local value of the function $\mathbf{T}^T_T$. Therefore, if at a particular point within the distribution the value of the function $\mathbf{T}^T_T$ gets very close to its average $\left\langle \mathbf{T}^T_T \right\rangle$ the relation between $\mathbf{T}^R_R$ and $\mathbf{T}^T_T$ becomes

$$\mathbf{T}^R_R \approx -\frac{1}{3}\mathbf{T}^T_T \ - \frac{\mathcal{G}(T)}{R^3} \quad \text{with} \quad \sigma_{\mathbf{T}^T_T} \approx 0 \ . \tag{10}$$

Physical insight can be gained by considering equations (5) and (6) in their static limit and in terms of the hydrodynamic density, $\rho(R)$ and pressure, $P_r(R)$. In this case, a static spherically symmetric distribution leads to

$$\rho(R) = P_r(R) + \frac{2}{R}\int_0^R P_r\left(\bar{r}\right)\,\mathrm{d}\bar{r} + \frac{\mathcal{H}}{R}\,, \qquad \text{or} \tag{11}$$

$$P_r(R) = \rho(R) - \frac{2}{R^3}\int_0^R \bar{r}^2\rho(\bar{r})\,\mathrm{d}\bar{r} + \frac{\mathcal{B}}{R^3}\,; \tag{12}$$

with $\mathcal{H}$ and $\mathcal{B}$ are arbitrary integration constants. It is clear that in equations (11) and (12) a collective behavior on the physical variables $\rho(R)$ and $P_r(R)$ is also present, and could be interpreted as a nonlocal interrelation between the energy density, $\rho(R)$, and the hydrodynamic pressure, $P_r(R)$,within the fluid. Any change in the pressure (density) takes into account the effects of the variations of the energy density (pressure) within an entire volume.

Following the above line of reasoning we shall consider the static limit of the equation (9):

$$P_r(R) = \mathcal{P}(R) + 2\sigma_{\mathcal{P}(R)} + \frac{\mathcal{B}}{R^3}\,, \tag{13}$$

where we have set

$$\mathcal{P}(R) = \frac{1}{3}\rho(R) \qquad \text{and} \quad \sigma_{\mathcal{P}(R)} = \left(\frac{1}{3}\rho(R) - \frac{1}{3}\left\langle\rho\right\rangle_R\right) = \left(\mathcal{P}(R) - \bar{\mathcal{P}}(R)\right), \tag{14}$$

with

$$\left\langle\rho\right\rangle_R = \frac{1}{\frac{4\pi}{3}R^3}\int_0^R 4\pi\bar{r}^2\rho(\bar{r})\,\mathrm{d}\bar{r} = \frac{M(R)}{\frac{4\pi}{3}R^3}. \tag{15}$$

In this limit the nonlocal integral term in (12) represents some kind of "average density" over the enclosed volume and, as far as the value of this nonlocal contribution gets closer to the value of the local density, the equation of state of the material becomes similar to the typical radiation dominated environment, i.e. $P_r(R) \approx \mathcal{P}(R) \equiv \frac{1}{3}\rho(R)$.

It is in the above sense, that we are going to refer to (5) or to (6), as *Nonlocal Equation of State* (*NLES* from now on) between the components $\mathbf{T}_T^T$ and $\mathbf{T}_R^R$ of the corresponding energy momentum tensor.

Nowadays, the relevance of nonlocal outcomes on the mechanical properties of fluids and materials are well known. There are many situations of common occurrence wherein nonlocal effects dominate the macroscopic behavior of matter in Modern Classical Continuum Mechanics and Fluid Dynamics. Examples coming from a wide variety of classical areas such as: damage and cracking analysis of materials, surface phenomena between two liquids or two phases, mechanics of liquid crystals, blood flow, dynamics of colloidal suspensions seem to demand more sophisticated continuum theories which can take into account different nonlocal effects. This is a very active area in recent material and fluid science and engineering (see [3] and references therein). Moreover, in radiating fluids when the medium is very transparent and photons escape efficiently, occupation numbers of atomic levels are, in general, no longer predicted by equilibrium statistical mechanics for local values of the temperature and density. In this case, the local state of a fluids is coupled by photon exchange to the state of the material within an entire interaction volume. These types of fluids are considered to be in *non-Local Thermodynamical Equilibrium* or non-LTE [4].

In this work only collapsing radiating configuration in General Relativity will be studied. Static models consistent with *NLES* require a detail analysis and will be considered elsewhere [5]. In what follows, we shall explore four major questions concerning the *NLES* in collapsing spherical sources in General Relativity, which can be stated as follows:

- Can matter configurations having *NLES* be associated with a particular symmetry of the space time and under what conditions ?
- Does the energy-momentum tensor obtained from (1) through the Einstein Equations, and satisfying (5), describe a physically reasonable anisotropic fluid ?
- Can this fluid represent a bounded gravitational source ?
- What types of gravitational collapse scenarios will emerge from bounded matter configurations with *NLES* ?

The structure of the present paper is guided by the above four questions. In the next section it is found that a solution of the Einstein equations, obtained from (1) restricted by (2), admits a Conformal Killing Vector. In Section III it is shown how anisotropic radiating configurations with *NLES* satisfy energy conditions (weak, strong and dominant) for imperfect fluids. The third question is addressed in Section IV. There we consider the consequences on the evolution of the boundary surface when this interior solution is matched to Vaidya exterior metric. Answers to the last question are explored in sections IV and V by studying the evolution of models of radiating spheres with a nonlocal equation of state (either 5 or 6). Finally in the last section our conclusions and results are summarized. If it is not explicitly stated as C(T) (or C(u), depending on the tetrad) we shall assume in this paper C as a constant parameter in the above equation (2).

II. SYMMETRIES AND *NLES* FOR AN ANISOTROPIC RADIATING SPHERE

It is well known that there are, at least, two reasons why it is interesting to start investigating the geometric properties of a particular space time. The first is a practical one: the formidable difficulties encountered in the study of the Einstein Field Equations, due to their non-linearity, can be partially overcame if some symmetry properties of the space-time are assumed. The second one is more fundamental; symmetries may also provide important insight and information into the general properties of self-gravitating matter configurations [6]. Guided by the above motivations we shall show, in this section, that the space times (1) having a *NLES* (5 or 6) admit a Conformal Motion (*CM*) with C considered as a constant parameter.

In general, *CM* is a map $M \to M$ such that the metric g of the space time transforms under the rule

$$g \to \tilde{g} = \mathrm{e}^{2\psi} g, \qquad \text{with} \qquad \psi = \psi(x^a).$$

This can be expressed as

$$\pounds_\xi \, \mathbf{g}_{ab} = \boldsymbol{\xi}_{a;b} + \boldsymbol{\xi}_{b;a} = \psi(x^\mu) \, \mathbf{g}_{ab} \, , \tag{16}$$

where $\boldsymbol{\xi}^\mu$ is called a Conformal Killing Vector (*CKV*). It is clear that other important symmetries as homotetic motions or self-similarity ($\pounds_\xi \; g_{ab} = 2\sigma \, g_{ab}$ with $\sigma = const.$) and isometries ($\pounds_\xi \; g_{ab} = 0$) are particular cases of *CM*. This symmetry imposes an important restriction on the hydrodynamic variables and consequently it is possible to obtain an equation of state for a space time (1) having a *CKV* [7–10].For the sake of simplicity and in order to solve the above Conformal Killing Equations (16) we shall assume C as a constant parameter. The most general case, $\mathrm{C} = \mathrm{C}(T)$, will be briefly considered at the end of the present work.

Additionally, we further restrict our calculations to a vector field $\boldsymbol{\xi}^\mu$ of the form

$$\boldsymbol{\xi}^\mu = \sigma(T, R)\delta^\mu_0 + \lambda(T, R)\delta^\mu_1 \, . \tag{17}$$

From (1) and (17), (16), with the condition (2), we have

$$\dot{\sigma} + \left[\beta' - \frac{1}{R}\right] \lambda + \sigma\dot{\beta} = 0 \, , \tag{18}$$

$$-\dot{\lambda} + \mathrm{C}^2\sigma' = 0 \, , \qquad \text{and} \tag{19}$$

$$\lambda' + \left[\beta' - \frac{1}{R}\right] \lambda + \sigma\dot{\beta} = 0 \, . \tag{20}$$

where dots and primes denote differentiation with respect to T and R, respectively.

Using (18) and (20) we get

$$-\ddot{\lambda} + \mathrm{C}^2\lambda'' = 0 \, . \tag{21}$$

A similar expression can be obtained for σ. The solutions of equation (21) and the corresponding equation for σ are

$$\lambda = f(\mathrm{u}) + g(\mathrm{v}) \, , \qquad \text{and} \qquad \sigma = m(\mathrm{u}) + n(\mathrm{v}) \, , \tag{22}$$

$f(\mathrm{u})$, $g(\mathrm{v})$, $m(\mathrm{u})$ and $n(\mathrm{v})$ being arbitrary functions of their arguments

$$\mathrm{u} = \mathrm{C}T + R \, , \qquad \text{and} \qquad \mathrm{v} = \mathrm{C}T - R \, . \tag{23}$$

With equations (22) in (19), we obtain

$$\sigma = \frac{1}{\mathrm{C}} \left[f(\mathrm{u}) - g(\mathrm{v}) \right] \, . \tag{24}$$

and, form (22) and (24), equation (20) can be written as

$$2\left[\frac{\partial\beta}{\partial \mathrm{u}} - \frac{1}{\mathrm{u} - \mathrm{v}}\right] f + \frac{\partial f}{\partial \mathrm{u}} - 2\left[\frac{\partial\beta}{\partial \mathrm{v}} + \frac{1}{\mathrm{u} - \mathrm{v}}\right] g - \frac{\partial g}{\partial \mathrm{v}} = 0 \, . \tag{25}$$

This can be integrated by demanding that the coefficient of $f(\mathrm{u})$ be only a function of u as well as the coefficient $g(\mathrm{v})$ a function of v:

$$\frac{\partial\alpha(\mathrm{u})}{\partial\mathrm{u}} \equiv \frac{\partial\beta}{\partial\mathrm{u}} - \frac{1}{\mathrm{u}-\mathrm{v}}\,, \qquad \text{and} \qquad \frac{\partial\delta(\mathrm{v})}{\partial\mathrm{v}} \equiv \frac{\partial\beta}{\partial\mathrm{v}} + \frac{1}{\mathrm{u}-\mathrm{v}}\,. \tag{26}$$

Thus, we get

$$\beta(\mathrm{u},\mathrm{v}) = \ln(\mathrm{u}-\mathrm{v}) + \alpha(\mathrm{u}) + \delta(\mathrm{v}) + C_1\,, \tag{27}$$

and equation (25) may now be written as

$$2\left(\frac{\partial\alpha}{\partial\mathrm{u}}\right) f + \frac{\partial f}{\partial\mathrm{u}} = 2\left(\frac{\partial\delta}{\partial\mathrm{v}}\right) g + \frac{\partial g}{\partial\mathrm{v}} = k\,; \tag{28}$$

where k and C_1 are constants of integration. This expression is integrated to give

$$f(\mathrm{u}) = e^{-2\,\alpha}\left[k\int e^{2\,\alpha} d\mathrm{u} + C_2\right] \qquad \text{and} \qquad g(\mathrm{v}) = e^{-2\,\delta}\left[k\int e^{2\,\delta} d\mathrm{v} + C_3\right]\,, \tag{29}$$

again, C_2 and C_3 are constants of integration. Then, (29) implies

$$\boldsymbol{\xi}^{\mu} = \frac{1}{\mathrm{C}}\left[f(\mathrm{u}) - g(\mathrm{v})\right]\,\delta^{\mu}_{0} + \left[f(\mathrm{u}) + g(\mathrm{v})\right]\,\delta^{\mu}_{1} \qquad \text{and} \qquad \psi = \frac{1}{R}\left[f(\mathrm{u}) + g(\mathrm{v})\right]\,, \tag{30}$$

and (27) may now be written as

$$\beta(T,R) = \ln(2kR) + \alpha(\mathrm{C}T + R) + \delta(\mathrm{C}T - R) + C_1\,. \tag{31}$$

Finally, the metric (1) reads:

$$ds^2 = R^2\left[4k^2 e^{2(\alpha+\delta)}\left(\mathrm{C}\,dT^2 - \frac{dR^2}{\mathrm{C}}\right) - d\Omega^2\right]\,. \tag{32}$$

It should be stressed that the above results have been obtained assuming (26) and C as a constant parameter in (2).

III. ENERGY CONDITIONS FOR IMPERFECT FLUIDS

We have found that in static configurations *NLES* are more easily handled within anisotropic fluids [5]. Thus, in order to explore how physically reasonable is the previous non-static fluid with a *NLES* and having a *CKV,* we shall study the energy conditions for an imperfect anisotropic fluid (unequal stresses, i.e. $P_r \neq P_\perp$ and heat flux). Although a pascalian assumption ($P_r = P_\perp$) is supported by solid observational ground, an increasing amount of theoretical evidence strongly suggests that, for certain density ranges, a variety of very interesting physical phenomena may take place giving rise to local anisotropy (see [11] and references therein).

For these fluids the energy-momentum tensor takes the form

$$\mathbf{T}_{\mu\nu} = (\rho + P_\perp)\mathbf{u}_\mu\mathbf{u}_\nu - P_\perp\mathbf{g}_{\mu\nu} + (P_r - P_\perp)\mathbf{n}_\mu\mathbf{n}_\nu + \mathbf{f}_\mu\mathbf{u}_\nu + \mathbf{f}_\nu\mathbf{u}_\mu\,, \tag{33}$$

where

$$\mathbf{u}_\mu = 4R^2 e^{2(\alpha+\delta)}\mathrm{C}\;\delta^{0}_{\mu}\,, \qquad \mathbf{n}_\mu = \frac{4R^2}{\mathrm{C}} e^{2(\alpha+\delta)}\;\delta^{1}_{\mu} \qquad \text{and} \qquad \mathbf{f}_\mu = -q\,\mathbf{n}_\mu\,, \tag{34}$$

with ρ, P_r, $P_\perp$ and q, denoting the energy density, the radial pressure, the tangential pressure and the heat flow ($\mathbf{f}_\mu\mathbf{f}^\mu = -q^2$), respectively. The fluid four-velocity is represented by $\mathbf{u}_\mu$ and $\mathbf{n}_\mu$ is a unit vector pointing the flux direction.

The energy conditions are obtained from the eigenvalues of the energy-momentum tensor (33), i.e. they emerge from the roots of the characteristic equation [12]

$$|\;\mathbf{T}^{b}_{a} - \lambda\,\boldsymbol{\delta}^{b}_{a}\;| = 0\,. \tag{35}$$

The four eigenvalues take the form

$$\lambda_0 = \frac{1}{2}\left[\rho - P_r + \Delta\right], \qquad \lambda_1 = \frac{1}{2}\left[\rho - P_r - \Delta\right], \qquad \text{and} \qquad \lambda_2 = \lambda_3 = -P_\perp, \tag{36}$$

where

$$\Delta = \left[(\rho + P_r)^2 - 4q^2\right]^{1/2}. \tag{37}$$

With the metric (32), Einstein's field equations can be written as

$$8\pi\,\mathbf{T}^T_T = 8\pi\,\rho = \frac{1}{R^2} + \frac{\mathcal{Z}}{R^4}\left[1 + 2\,R\,(\alpha' - \delta')\right], \tag{38}$$

$$8\pi\,\mathbf{T}^R_R = -8\pi\,P_r = \frac{1}{R^2} - \frac{\mathcal{Z}}{R^4}\left[3 + \quad 2R\,(\alpha' - \delta')\right], \tag{39}$$

$$8\pi\,\mathbf{T}^\Theta_\Theta = -8\pi\,P_\perp = \frac{\mathcal{Z}}{R^4}, \qquad \text{and} \tag{40}$$

$$8\pi\,\mathbf{T}_{TR} = -8\pi\,q = \frac{2\mathcal{Z}}{R^3}\left(\dot{\alpha} + \dot{\delta}\right), \tag{41}$$

where

$$\mathcal{Z} \equiv \frac{\mathrm{C}}{4e^{2(\alpha+\delta)}} > 0\,. \tag{42}$$

From two of the above Einstein equations, (38) and (39), we find that the conditions for energy density and radial pressure to be positive are

$$\alpha' - \delta' \;\geq\; -\frac{R}{2\mathcal{Z}} - \frac{1}{2R} \qquad \text{and} \qquad \alpha' - \delta' \;\geq\; \frac{R}{2\mathcal{Z}} - \frac{3}{2R}\,. \tag{43}$$

Considering (40) and (42) it can easily be shown that the tangential pressure, $P_\perp$, is negative. Consequently the condition (37) is fulfilled if

$$2P_\perp + \Delta \geq 0 \quad \Rightarrow \quad \alpha' - \delta' \;\geq\; -\left(\dot{\alpha} + \dot{\delta}\right) - \frac{1}{R}\,. \tag{44}$$

We then find that the weak energy condition is achieved if:

$$\rho - P_r + \Delta \geq 0 \quad \Rightarrow \quad \Delta \geq \frac{2\mathcal{Z}}{R^4} - \frac{2}{R^2}\,, \tag{45}$$

the dominant energy condition is held if

$$\rho - P_r \geq 0 \Rightarrow \frac{\mathcal{Z}}{R^2} \leq 1 \qquad \text{and} \qquad \rho - P_r - 2P_\perp + \Delta \geq 0 \;\Rightarrow \Delta \geq\; -\frac{2}{R^2}\,, \tag{46}$$

and the strong energy condition is accomplished if

$$2P_\perp + \Delta \geq 0 \Rightarrow \Delta \geq \frac{2\mathcal{Z}}{R^4}\,. \tag{47}$$

Next, we are going to show that it is possible to fulfill simultaneously all of the above energy conditions. From the definition for $\mathcal{Z}$ and (46) it is now seen that

$$0 < \frac{\mathcal{Z}}{R^2} \leq 1\,. \tag{48}$$

Therefore, inequalities (45), (46) and (47) yield

$$\frac{\Delta R^2}{2} \geq \frac{\mathcal{Z}}{R^2} \Rightarrow e^{2(\alpha+\delta)} \geq \frac{\mathrm{C}}{2\Delta R^4}. \tag{49}$$

Now, considering (48), (43) and (44) we have

$$\alpha' - \delta' \geq -\frac{1}{R} \qquad \text{and} \qquad \dot{\alpha} + \dot{\delta} \geq 0\,. \tag{50}$$

By using equation (31) we can also see that

$$\beta' = \frac{1}{R} + \alpha' - \delta' \,, \tag{51}$$

which, with the help of (50) becomes

$$\beta' \geq 0 \,. \tag{52}$$

This is a well known restriction on the metric function β [13]. Therefore, it is clear that matter configurations satisfying a *NLES* and having a *CKV* can represent reasonable fluids in General Relativity. Next section will be devoted to study the consequences of matching this type of fluids to an exterior solution.

IV. JUNCTION CONDITIONS

In order to address the third question, we shall demand that the line element (32) join the Vaidya exterior metric at the boundary of the configuration. We shall also recall that two regions of a space-time match across a separating hypersurface S if the first and the second fundamental forms are continuous across this boundary surface S [14].

The Vaidya line element outside the source can be written as

$$ds^2_{(+)} = H du^2 + 2 du dr - r^2 d\Omega^2 \,, \qquad \text{with} \qquad H = 1 - \frac{2M\,(u)}{r} \,, \tag{53}$$

where the subscript $(+)$ refers to the exterior region of the space time. In these coordinates the boundary equation takes the form: $r = r_s(u)$, where the subscript s reminds that the corresponding function has been evaluated at boundary of the configuration. Then, the induced metric on the boundary surface, from the outside, is

$$\left(ds^2\right)^{(+)}_s = \left[H_s + 2\frac{dr_s}{du}\right] du^2 - r_s^2 d\Omega^2 \,, \tag{54}$$

and the line element on the boundary, from the inside, reads

$$\left(ds^2\right)^{(-)}_s = 4R_s^2 e^{2(\alpha_s+\delta_s)} \left[\mathrm{C} - \frac{1}{\mathrm{C}}\left(\frac{dR_s}{dT}\right)^2\right] dT^2 - R_s^2 d\Omega^2 \,, \tag{55}$$

where we have used the fact that the equation of the boundary in the coordinates (T, R, Θ, Φ) reads: $R = R_s(T)$. Demanding continuity of the first fundamental form, we get at once

$$2R_s e^{(\alpha_s+\delta_s)} \left[\mathrm{C} - \frac{1}{\mathrm{C}}\left(\frac{dR_s}{dT}\right)^2\right]^{1/2} dT = \left[H_s + 2\frac{dr_s}{du}\right]^{1/2} du \,, \qquad \text{and} \tag{56}$$

$$R_s\,(T) \;= r_s\,(u) \;. \tag{57}$$

It can be shown [15] that (56) is equivalent to

$$e^{2(\alpha_s+\delta_s)} = \frac{1}{4R_s^2} \;. \tag{58}$$

Also, a straighforward calculation shows that the continuity of the second fundamental form across the boundary surface is equivalent to the condition [14]

$$[q]_s = [P_r]_s \;. \tag{59}$$

Using (39) and (41), the equation (59) gives the evolution of the boundary surface:

$$R_s = \frac{\kappa}{2} \frac{1}{\dot{\alpha_s} + \dot{\delta_s} + \alpha'_s - \delta'_s} \,, \tag{60}$$

where the constant κ is :

$$\kappa \equiv \frac{1-3\mathrm{C}}{\mathrm{C}} \ . \tag{61}$$

Now, substituting equation (60) in (58), we obtain

$$\dot{\alpha_s} + \dot{\delta_s} + \alpha_s' - \delta_s' = \kappa \, e^{\alpha_s + \delta_s} \ . \tag{62}$$

It is useful to write equation (62) in terms of the variables u and v

$$\left(\frac{\partial \alpha_s}{\partial \mathrm{u}} + \frac{\partial \delta_s}{\partial \mathrm{v}} \right) (\mathrm{C}+1) = \kappa \, e^{(\alpha_s(\mathrm{u}) + \delta_s(\mathrm{v}))} \ , \tag{63}$$

which can be easily integrated for constant α_s. Thus, assuming $\alpha_s = K = const$, we get

$$\delta_s = -K - \ln \left[-\frac{\kappa \, (\mathrm{v} - (\mathrm{C}+1) \, C_2)}{\mathrm{C}+1} \right] \ . \tag{64}$$

Next, considering K and δ_s in equation (58), the evolution of the boundary surface can be written as

$$R_s = \frac{\kappa \, [\mathrm{C}T - (\mathrm{C}+1) \, C_2]}{\kappa \pm 2 \, (\mathrm{C}+1)} \ , \tag{65}$$

where C_2 is a constant of integration. It is clear that there are two possible evolutions for the bounding surface and in both of them R_s is a linear function of the time coordinate, namely

$$R_s^+ = \frac{(1-3\mathrm{C})\,\mathrm{C}}{2\mathrm{C}^2 - \mathrm{C} + 1} \, T - \frac{(1-3\mathrm{C})\,(\mathrm{C}+1)\, C_2^+}{2\mathrm{C}^2 - \mathrm{C} + 1} \ , \qquad \text{and} \tag{66}$$

$$R_s^- = \frac{(1-3\mathrm{C})\,\mathrm{C}}{-2\mathrm{C}^2 - 5\mathrm{C} + 1} \, T - \frac{(1-3\mathrm{C})\,(\mathrm{C}+1)\, C_2^-}{-2\mathrm{C}^2 - 5\mathrm{C} + 1} \ . \tag{67}$$

Notice that expression (61) for the constant κ has been used and we have denoted C_2^+ and C_2^- the corresponding constants for the two equations emerging from (65).

Let us consider each of the above solutions separately. First, it is clear that in (66), due to

$$2\mathrm{C}^2 - \mathrm{C} + 1 > 0 \qquad \forall \ \mathrm{C} \ \in \ (0,1) \ , \tag{68}$$

expanding configurations can be obtained if $\mathrm{C} \ \in \ \left(0 \ , \ \frac{1}{3}\right)$ and collapsing ones for $\mathrm{C} \ \in \ \left(\ \frac{1}{3} \ , \ 1\right)$. Additionally, because $R_s^{\pm} > 0 \quad \forall \ T \ \in \ (0, \infty)$, several values of the possible initial configurations (M_0 and $R_s^{\pm}$) have to be excluded. Concerning the first case, R_s^+, the allowed values for the constant C are

$$\mathrm{C} \ \in \ \left(0 \ , \ \frac{1}{3}\right) \qquad \Rightarrow \ \text{Expanding Configurations,} \qquad \text{and} \tag{69}$$

$$\mathrm{C} \ \in \ \left(\frac{1}{3} \ , \ 1\right) \qquad \Rightarrow \ \ \text{Contracting Configurations.} \tag{70}$$

In the second case (equation (67)), the permitted values can be expressed as

$$\mathrm{C} \ \in \ \left(0 \ , \ \frac{1}{4}\left(\sqrt{33} - 5\right)\right) \cup \ \left(\frac{1}{3} \ , \ 1\right) \qquad \Rightarrow \ \text{Expanding Configurations,} \qquad \text{and} \tag{71}$$

$$\mathrm{C} \ \in \ \left(\frac{1}{4}\left(\sqrt{33} - 5\right) \ , \ \frac{1}{3}\right) \qquad \Rightarrow \ \ \text{Contracting Configurations.} \tag{72}$$

It is clear that C_2^+ and C_2^- can be solved from (66), and (67), namely

$$C_2^+ = -\frac{R_{so}^+ \left(2\,\mathrm{C}^2 - \mathrm{C} + 1\right)}{(1 - 3\,\mathrm{C})\,(\mathrm{C}+1)} \ , \qquad \text{and} \tag{73}$$

$$C_2^- = \frac{R_{so}^- \left(2\,\mathrm{C}^2 + 5\,\mathrm{C} - 1\right)}{(1 - 3\,\mathrm{C})\,(\mathrm{C}+1)} \ , \tag{74}$$

with R^+_{so} and R^-_{so} the initial values for R^+_s and R^-_s, respectively. For the present work we shall only consider those values of C that induce collapsing configurations.

For this particular matter distribution, starting with an initial radius, $R^{\pm}_{so}$, and mass, M_0, we shall define the *Total Evolution Time*, Tev, as the time it takes to radiate away its total initial mass M_0. Easily, we get the expressions for the Tev corresponding to (66), and (67), concerning the values of the constant parameter C in (70) and (72), respectively. Thus, they can be written as

$$Tev_I = \frac{R^+_{so}\left(1-3\,\frac{M_0}{R^+_{so}}+\left(\frac{2M_0}{R^+_{so}}\right)^2\right)}{\left(1-3\,\frac{M_0}{R^+_{so}}\right)\left(1-2\frac{M_0}{R^+_{so}}\right)}\,, \qquad \text{and} \tag{75}$$

$$Tev_{II} = -\frac{R^-_{so}\left(3\,-9\,\frac{M_0}{R^-_{so}}+\left(\frac{2M_0}{R^-_{so}}\right)^2\right)}{\left(1-3\,\frac{M_0}{R^-_{so}}\right)\left(1-2\frac{M_0}{R^-_{so}}\right)}\,. \tag{76}$$

It is clear that given the initial values for the mass and radius (therefore, the parameter $\mathrm{C}=\frac{R^{\pm}_{so}}{M_0}$), Tev for both evolutions is fully determined. It is also worth mentioning the particular astrophysical scenario that the evolution of these matter configurations could simulate. Both of them (either (66), or (67)) start at a particular value of the parameter C, completely radiate away their initial mass and disappear leaving no remnant. In this sense this scenario resembles Type I Supernova but, in the present case, it begins from a neutron star progenitor (characteristic values for the radius, $R^{\pm}_{so}$ and the mass M_0) and gravitation plays an important role. In the standard Type I Supernova picture there is a white dwarf progenitor and gravitation is not longer the main cause of the emission processes.

In Figure 1 the total evolution times Tev_I and Tev_{II} are sketched as a function of the initial mass. In both cases we have assumed 10 Kms. as the value for the initial radius. This value can be considered as typical for neutron stars [1,2]. It can be appreciated from this figure that as the initial mass M_0 rises, the total time, Tev_I, increases asymptotically up to the value $M_0=\frac{R^-_{so}}{3}$. The other branch Tev_{II} decreases coming from this same asymptotic maximum for M_0. These two branches of the evolution time may represent different astrophysical phenomena. In first case, $M_0<\frac{R^-_{so}}{3}$, it is apparent that the more massive the configuration is, the more time it takes to radiate the initial mass. In the second branch, $M_0>\frac{R^+_{so}}{3}$, more massive spheres lead to more violent collapses. The value of $M_0=\frac{R^-_{so}}{3}$ emerges as a critical point of stability for a configuration with a *NLES*, having a *CKV* and matched to a Vaidya exterior solution. Any perturbation accreting over (or expelling mass from) this type of relativistic sphere will initiate processes which radiate away a significant amount of energy.

In the next section we shall consider the hydrodynamic consequences of configuration satisfying a *NLES* for both cases: C =*const.* and C = C(T).

V. COLLAPSING RADIATING SPHERES

Finally, to determine if anisotropic fluid spheres with a *NLES* can conform a feasible matter candidate to stellar models, we shall study collapsing scenarios for radiating configuration of these type of materials. In what follows, radiating collapsing models will be matched to the Vaidya exterior metric and energy density will always be positive and larger than pressure everywhere within the fluid distribution. Finally, the fluid velocity, as measured by the locally Minkowskian observer, will always be less than one.

Besides these "regularity" conditions, the modeling emerges from a heuristic assumption relating density, pressure and radial matter velocity. This *ansatz*, guided by solid physical principles, allows the generation of radiating solutions from a known static "seed" solutions and reduces the problem of solving Einstein Equations to a numerical integration of a system of ordinary differential equations for quantities evaluated at the surfaces (shocks and/or boundaries). The rationale behind this *ansatz* can be grasped in terms of the characteristic times for different processes involved in a collapse scenario. If the hydrostatic time scale, $\mathcal{T}_{HYDR}$, which is of the order $\sim 1/\sqrt{G\rho}$ (where G is the gravitational constant and ρ denotes the mean density), is much smaller than the *Kelvin-Helmholtz* time scale ($\mathcal{T}_{KH}$), then in a first approximation, the inertial terms in the equation of motion can be ignored [16].

A. The metric and the matter

The method we use to build these models is a general strategy presented several years ago by L. Herrera, J. Jiménez and G. Ruggeri (HJR). Only a very brief description of this method is given here. We refer the reader to [17] and [18] for details. In this method a nonstatic spherically symmetric distribution of matter is assumed. The metric representing this distribution, in Bondi radiation coordinates [19], takes the form

$$ds^2 = e^{4\beta(u,r)}h\,(u,r)\;\,du^2 + 2e^{2\beta(u,r)}\;du\,dr - r^2(d\vartheta^2 + \sin^2\phi\,d\phi^2)\,, \qquad \text{with} \quad h \equiv 1 - \frac{2m\,(u,r)}{r}. \tag{77}$$

Here $u = x^0$ is a time like coordinate, $r = x^1$ is the null coordinate and $\theta = x^2$ and $\phi = x^3$ are the usual angle coordinates. The u -coordinate is the retarded time in flat space-time and, therefore, u -constant surfaces are null cones open to the future. The function $m\,(u,r)$ is the generalization, inside of the distribution, of the "mass aspect" defined by Bondi [19] which in the static limit coincides with the Schwarzschild mass. In addition we consider an anisotropic fluid sphere composed by a material medium plus radiation. For a observer with a radial velocity $-\omega$ with respect to the fluid, the energy-momentum tensor can be written as [18,20]

$$\mathbf{T}_{\mu\nu} = (\rho + P_\perp)\mathbf{u}_\mu\mathbf{u}_\nu - P_\perp\mathbf{g}_{\mu\nu} + (P_r - P_\perp)\mathbf{n}_\mu\mathbf{n}_\nu + \mathbf{f}_\mu\mathbf{u}_\nu + \mathbf{f}_\nu\mathbf{u}_\mu\,, \tag{78}$$

with

$$\mathbf{u}_\mu = \frac{1}{(1-\omega^2)^{\frac{1}{2}}}\left[h^{\frac{1}{2}}e^{2\beta}\delta^0_\mu + \frac{1-\omega}{h^{\frac{1}{2}}}\delta^1_\mu\right]\,, \quad \mathbf{n}_\mu = \frac{1}{(1-\omega^2)^{\frac{1}{2}}}\left[-\omega\delta^0_\mu + \frac{1-\omega}{h^{\frac{1}{2}}}\delta^1_\mu\right]\,, \quad \mathbf{f}_\mu = -q\,\mathbf{n}_\mu\,, \tag{79}$$

and where the physical variables, as measured by a local *minkowskian* observer, are represented by the energy density, ρ; the radial and tangential pressure, P_r, $P_\perp$, respectively; the radiation energy flux density $\mathbf{f}_\mu$ and fluid radial velocity, $-\omega$. The co-moving *minkowskian* observer coincides with the *Lagrangean frame* (the *proper frame*) where the interaction between radiation and matter can be easily handled [4]. Notice that the radiation is treated in the diffusive regime, therefore it is considered to have a mean free path much smaller than the characteristic length of the system. Within this regime, radiation is locally isotropic and we have

$$\left.\begin{array}{c}\rho_R = 3\mathcal{P}\\ \mathcal{F} = q\end{array}\right\} \Longrightarrow P_r = \hat{P} + \mathcal{P}\,. \tag{80}$$

Where, $\hat{P}$, describes the fluid radial pressure and the radiation contribution to the energy density, energy flux density and radial pressure are represented by ρ_R, $\mathcal{F}$ and $\mathcal{P}$, respectively. It is clear that in this radiation limit the total radial pressure, P_r, encompasses the hydrodynamic and the radiation contribution to the pressure (see [21] for details).

B. The Einstein Field Equations and the Matching

Inside the matter distribution the Einstein field equations can be written as

$$\frac{\rho + 2\omega q + \omega^2 P_r}{1-\omega^2} - \frac{1}{4\pi r^2}\left(-\frac{\dot{m}}{h\,e^{2\beta}} + m'\right) = 0\,, \tag{81}$$

$$\frac{\rho - \omega P_r - (1-\omega)q}{1+\omega} - \frac{1}{4\pi r^2}m' = 0\,, \tag{82}$$

$$\frac{(\rho + P_r - 2q)(1-\omega)}{1+\omega} - \frac{h}{2\pi r}\beta' = 0\,, \qquad \text{and} \tag{83}$$

$$P_\perp + \frac{\dot{\beta}'}{4\pi e^{2\beta}} - \frac{h}{8\pi}\left(2\beta'' + 4\,(\beta')^2 - \frac{\beta'}{r}\right) - \frac{3\beta'(1-2m') - m''}{8\pi r} = 0\,. \tag{84}$$

As was stressed by Bondi [13] it is possible to solve algebraically the physical variables of equations (81) - (84) if we have both $m\,(u,r)$ and $\beta\,(u,r)$ (and their derivatives) plus another equation that relates the tangential and radial pressure. This method will be clarified by an example in the next section.

From the field equations (82) and (83) it is clear that they can be (formally) integrated yielding

$$m = 4\pi \int_0^r \frac{\rho - \omega P_r - (1-\omega)q}{1+\omega} r^2 dr = 4\pi \int_0^r r^2 \tilde{\rho} dr\,, \qquad \text{and} \tag{85}$$

$$\beta = 2\pi \int_a^r \frac{(\rho + P_r - 2q)(1-\omega)}{1+\omega} \frac{r}{h} dr = 2\pi \int_a^r \left(\tilde{\rho} + \tilde{P}\right) \frac{r}{h} dr\,, \tag{86}$$

where

$$\tilde{\rho} \equiv \frac{\rho - \omega P_r - (1-\omega)q}{1+\omega}\,, \qquad \text{and} \qquad \tilde{P} \equiv \frac{-\omega\rho + P_r - (1-\omega)q}{1+\omega}\,. \tag{87}$$

These auxiliary functions $\tilde{\rho}$ and $\tilde{P}$ must coincide with the energy density and the radial pressure in the static limit.

Matching the Vaidya metric to the Bondi metric (77) at the surface of the fluid distribution, $r = a$, implies $\beta_a = 0$ with the continuity of the mass function $m(u, r)$, i.e. the continuity of the first fundamental form. The continuity of the second fundamental form leads to (again, see [21] for details)

$$\tilde{P}_a = -\omega_a \tilde{\rho}_a\,. \tag{88}$$

The subscript a indicates that the corresponding quantity is evaluated at the boundary surface.

C. Surface equations

As we have mentioned, the crucial point of the HJR method is the assumption that the radial dependence of $\tilde{\rho}$ and $\tilde{P}$ can be borrowed from a "seed" static solution and, therefore the metric functions $m(u, r)$ and $\beta(u, r)$ can be determined from eq. (85) and (86), up to some functions of the time-like coordinate u, which in turn are computed by integrating a system of ordinary differential equation coming from the boundary conditions. This assumption represents a correction to the first approximation in the radial velocity (i.e., quasi-stationary approximation) and is expected to yield good results whenever $\mathcal{T}_{KH} \gg \mathcal{T}_{HYDR}$. Fortunately enough, this condition is fulfilled for almost all kind of stellar objects. For example, in the case of the Sun we get $\mathcal{T}_{KH} \sim 107$ years, whereas $\mathcal{T}_{HYDR} \sim 27$ minutes. Also, the *Kelvin-Helmholtz* phase of the birth of a neutron star last for about tens of seconds, whereas for a neutron star of one solar mass and a ten kilometer radius, we obtain $\mathcal{T}_{HYDR} \sim 10^{-4} sec$.

It is clear that with the metric functions completely determined, the physical variables (ρ, P_r, $P_\perp$ $\mathbf{f}_\mu$, and ω) are algebraically solved from the field equations (81) through (84).

Scaling the radius a, the total mass m_a and the timelike coordinate u by the total initial, mass $m_a(0)$, i.e.

$$A(u) \equiv \frac{a(u)}{m_a(0)}, \quad M(u) \equiv \frac{m_a(u)}{m_a(0)}, \qquad \text{and} \quad u \equiv \frac{u}{m_a(0)}\,, \tag{89}$$

and defining

$$F(u) \equiv 1 - \frac{2M(u)}{A(u)}, \qquad \text{and} \qquad \Omega(u) \equiv \frac{1}{1-\omega_a}\ . \tag{90}$$

The first surface equation comes from the definition of the velocity in radiation coordinates, as

$$\dot{A} = F(\Omega - 1)\,. \tag{91}$$

The second surface equation arises from the luminosity evaluated at the surface which can be written as

$$L = -\dot{M} = 4\pi A^2 q_a (2\Omega - 1) F\,. \tag{92}$$

Using equation (91) and the definitions (90), we can re-state equation (92) as

$$\dot{F} = \frac{2L + F(1-F)(\Omega - 1)}{A}\,. \tag{93}$$

Finally, some straightforward manipulations coming from the field equations (82), (82), (83) and (84), lead to

$$\left[-\frac{d}{du}\left(\frac{\tilde{\rho} + \tilde{P}}{h}\right)\right] + \tilde{R} - \frac{2}{r}\left(P_r - \tilde{P}\right) = 0\,, \tag{94}$$

where

$$\tilde{R} = \tilde{P}' + \left(\frac{\tilde{\rho}+\tilde{P}}{h}\right)\left(4\pi r\tilde{P} + \frac{m}{r^2}\right) - \frac{2}{r}\left(P_\perp - P_r\right) . \tag{95}$$

Notice that (94) corresponds to a generalization of the Tolman-Oppenheimer-Volkov equation for any dynamic radiative situation. Evaluating it on the boundary surface $r = a_-$ and using the definitions (90), the third surface equation can be obtained, i.e.

$$\frac{\dot{F}}{F} + \frac{\dot{\Omega}}{\Omega} - \frac{\dot{\tilde{\rho}}_a}{\tilde{\rho}_a} + \frac{F\Omega^2}{\tilde{\rho}_a}\tilde{R}_a - \frac{2F\Omega}{A}\frac{P_{ra}}{\tilde{\rho}_a} - G = 0\,, \tag{96}$$

where

$$G = (1-\Omega)\left[\frac{4\pi A\,(3\Omega-1)}{\Omega}\tilde{\rho}_a - \frac{3+F}{2A} + F\Omega\frac{\tilde{\rho}'_a}{\tilde{\rho}_a} + \frac{2F\Omega}{A\tilde{\rho}_a}\left(P_\perp - P_r\right)_a\right] . \tag{97}$$

Equations (91), (93) and (96) constitute the *System of Surface Equations* (*SSE*). This system can be integrated for any given radial dependence of the effective variables $\tilde{\rho}$ and $\tilde{P}$. In the context HJR with anisotropy a general equation has to be provided in order to relate the tangential and the radial pressure [22,18,20].

$$P_\perp\,(u,r) - P_r\,(u,r) = \frac{1}{2}\left[r\tilde{P}' + \frac{\tilde{\rho}+\tilde{P}}{1-\frac{2m}{r}}\left(\frac{m}{r} + 4\pi r^2\tilde{P}\right)\right] . \tag{98}$$

It is clear that if the anisotropic equation of state (98) is given, we have three differential equations to get four unknown functions, i.e. the exterior radius $A(u)$, the gravitational potential at the surface, $F(u)$, the radial velocity, $\Omega(u)$, and the luminosity profile, $L(u)$. Since the only observable quantity entering a "real" gravitational collapse is the luminosity, it seems reasonable to provide such a profile as an input for our modelling.

D. The HJR formalism and *NLES*

At this point we shall verify that, in the HJR formalism, models having (2), satisfy a nonlocal equation (5 or 6) within the distribution. Taking derivatives of (2) with respect to r, we get

$$-\frac{m'}{r} + \frac{m}{r^2} = -\beta' e^{-2\beta}\,\mathrm{C}(u)\,, \tag{99}$$

and using (2), (85) and (86) with (99) we obtain

$$m = 2\pi r^3\left(\tilde{\rho} - \tilde{P}\right) . \tag{100}$$

On the other hand, taking derivatives of (100) with respect to r and comparing it with (85) it follows that

$$\tilde{\rho} - 3\tilde{P} + r\left(\tilde{\rho}' - \tilde{P}'\right) = 0\,. \tag{101}$$

As expected, this expression emerges from the condition (2) and in terms of the field equations it reads

$$\mathbf{G}^u_u + 3\mathbf{G}^r_r + r\left[(\mathbf{G}^u_u)' + (\mathbf{G}^r_r)'\right] = 0\,. \tag{102}$$

It should be noticed that from equation (101) two expressions for *NLES* can be obtained. They can be written as

$$\tilde{\rho}(r,u) = \tilde{P}(r,u) + \frac{2}{r}\int_0^r \tilde{P}\,(\bar{r},u)\mathrm{d}\bar{r} + \frac{\mathcal{H}(u)}{r}\,, \qquad \text{or} \tag{103}$$

$$\tilde{P}(r,u) = \tilde{\rho}(r,u) - \frac{2}{r^3}\int_0^r \bar{r}^2\tilde{\rho}(\bar{r},u)\mathrm{d}\bar{r} + \frac{\mathcal{B}(u)}{r^3}\,. \tag{104}$$

In the present case, equations (103) and (104) are (5) and (6), respectively, expressed in terms of the effective variables defined in (87). From this new aspect of the *NLES*, it is clear that the contribution to the pressure and the energy

density coming from the radiation energy flux, as well as the effects of the fluid velocity, are taken into account in the nonlocal character of the equation of state. Moreover, equation (9), in terms of the effective variables, becomes

$$\tilde{P}(r,u)=\frac{1}{3}\tilde{\rho}(r,u)-2\left(\frac{1}{3}\tilde{\rho}(r,u)-\frac{1}{3}\left\langle\tilde{\rho}(r,u)\right\rangle\right)+\frac{\mathcal{B}(u)}{r^3}. \tag{105}$$

Clearly, it is the dynamic generalization of equation (13), with the corresponding interpretation of $\tilde{P}(r,u)=\frac{1}{3}\tilde{\rho}(r,u)$, as the "effective equation of state" which may be affected by the standard deviation

$$\sigma_{\frac{1}{3}\tilde{\rho}(r,u)}=\left(\frac{1}{3}\tilde{\rho}(r,u)-\frac{1}{3}\left\langle\tilde{\rho}(r,u)\right\rangle\right), \tag{106}$$

coming from the contribution of the nonlocal term $\langle\tilde{\rho}(r,u)\rangle$.

There is an other important consequence arising from (2) in terms of the surface variables in the HJR formalism. First, comparing the condition (2) with the definition for $F(u)$, it follows

$$h_a=1-2\frac{M}{A}=F(u)=\mathrm{C}(u). \tag{107}$$

Notice that $\mathrm{C}(u)$ is related to the gravitational potential at the surface. Next, evaluating (100) at the surface and using (88) we get

$$m_a=2\pi a^3\left(1+\omega_a\right)\tilde{\rho}_a\,. \tag{108}$$

This equation can be written as a function of the dimensionless variables (89) as

$$1-\mathrm{C}(u)=\frac{4\pi A(u)^2}{\Omega(u)}\left(2\Omega(u)-1\right)\tilde{\rho}_a(u)\,, \tag{109}$$

therefore, the "velocity" of the surface $\Omega(u)$ can be obtained

$$\Omega(u)=\frac{4\pi A(u)^2\tilde{\rho}_a(u)}{8\pi A(u)^2\tilde{\rho}_a(u)+\mathrm{C}(u)-1}\,. \tag{110}$$

It is clear that it relates the velocity and the effective density at the surface.

Now we are going to provide several "seed" solutions to explore reasonable collapsing scenarios compatible with a *NLES* in the HJR formalism.

E. Collapsing Scenarios

The first example considered as a starting static equation of state is the *Schwarzschild* solution. This model represents a generalization of a radiating incompressible anisotropic fluid of homogeneous density. The effective density can be written as

$$\tilde{\rho}_{sh}=\frac{1}{8\pi}f(u)\,. \tag{111}$$

With equation (85) evaluated at the surface we find $f(u)$ and using (110) we obtain that the fluid velocity at the surface is constant and takes the value $\omega_a=-\frac{1}{3}c$.

The second case of study is the anisotropic *Tolman-VI-like* model. Again, in the static limit this solution is not deprived of a physical meaning. The static *Tolman VI* solution approaches that of a highly relativistic Fermi Gas and, therefore, one with the corresponding adiabatic exponent of 4/3. In this case we have

$$\tilde{\rho}_{T_{VI}}=\frac{1}{8\pi r^2}g(u)\,. \tag{112}$$

Again equation (85), can be evaluated at the surface, with the restriction (110), leading to a constant velocity at the surface: $\omega_a=c$.

Although the above equations of state have been successfully worked out within HJR formalism, in our study with a *NLES* they lead to very restrictive matter configurations where the velocity of the surface is held constant during the collapse. It is worth mentioning that the above results for the *Schwarzschild-like* and *Tolman-VI-like* models are valid for the general case $\mathrm{C} = \mathrm{C}(u)$.

Next, we use as a "seed" another static equation of state which represents a richer description for ultracompact nuclear matter. We shall explore the consequences of adopting a static solution proposed by M. K. Gokhroo and A. L. Mehra in 1994 [23]. This solution corresponds to an anisotropic fluid with variable density. It has been used in the context of the HJR method to study several interesting properties of an anisotropic fluid [20] describing the Kelvin-Helmoltz phase in the birth of a neutron star [24,25]. It leads, under some circumstances [20], to densities and pressures given rise to an equation of state similar to the Bethe-Börner-Sato newtonian equation for nuclear matter [1,2].

The energy density and radial pressure for this static space-time are assumed to be

$$\rho\,(r) \;= \rho_c \left[1 - k\frac{r^2}{a^2}\right] \;, \quad (0 \leq k \leq 1) \;, \qquad \text{and} \tag{113}$$

$$P\,(r) \;= P_c \left(1 - \frac{2m\,(r)}{r}\right)\left(1 - \frac{r^2}{a^2}\right)^{n} \;, \qquad \text{with} \qquad n \geq 1\,; \tag{114}$$

where ρ_c, P_c and a are the energy density, radial pressure and the surface of the sphere, respectively. Finally, k is an arbitrary constant which may take values between zero and one and a constant parameter $n \geq 1$.

F. The modelling performed

In applying the HJR method to the Gokhroo and Mehra model [23], we assume that the effective variable $\tilde{\rho}$ has the same r dependence as $\rho\,(r)$, i.e.

$$\tilde{\rho}\,(u, r) = \tilde{\rho}_c\,(u)\left[1 - K(u)\frac{r^2}{a^2}\right] \;, \quad (0 \leq K\,(u) \leq 1) \;. \tag{115}$$

In the above equation, $K\,(u)$ is a function that may take values between zero and one only [20], and the energy density at the center of the configuration, $\tilde{\rho}_c\,(u)$, coincides with the ρ_c in the static case.

It is necessary to relate the unknown functions $K\,(u)$ and $\tilde{\rho}_c\,(u)$ to the surface functions $A(u)$, $\Omega(u)$, $F(u)$ and $L(u)$. Thus, we evaluate the expression (85) at the surface of the configuration:

$$m_a = M \Rightarrow \frac{4\pi A^3 \tilde{\rho}_c(u)}{15}\,(5 - 3K(u)) = \frac{A}{2}\,(1 - \mathrm{C}\,(u)) \;. \tag{116}$$

Therefore, from the above equation it is possible to find $\tilde{\rho}_c$:

$$\tilde{\rho}_c = \frac{15\,(1 - \mathrm{C}\,(u))}{8\pi A^2\,(5 - 3K(u))}\,, \tag{117}$$

and the expression for $K\,(u)$ is obtained from (110), namely

$$K\,(u) = \frac{5}{3}\,\frac{4\Omega - 3}{8\Omega - 5} = \frac{5}{3}\,\frac{1 + 3\,\omega_a}{3 + 5\,\omega_a}\,. \tag{118}$$

Thus, the effective energy density is written as

$$\tilde{\rho}(r, u) = \frac{[1 - \mathrm{C}\,(u)]\,r^2}{16\pi A^4\,(2\Omega - 1)}\left[5\,(3 - 4\Omega) + 3\frac{A^2}{r^2}\,(8\Omega - 5)\right] \;, \tag{119}$$

and from (104) the effective pressure, can be computed as

$$\tilde{P}(r, u) = \frac{[1 - \mathrm{C}\,(u)]\,r^2}{16\pi A^4\,(2\Omega - 1)}\left[3\,(3 - 4\Omega) + \frac{A^2}{r^2}\,(8\Omega - 5)\right] \;. \tag{120}$$

Notice that the function $\mathcal{B}(u)$ should vanished in order to fulfil the matching condition (88), i.e. $\mathcal{B}(u) = 0$, $\forall\, u$.

Finally, by virtue of (85) and (86) the metric functions read

$$m\left(u,r\right) = \frac{r^3\left(1-\mathrm{C}\left(u\right)\right)}{4A^4\left(2\Omega-1\right)}\left[r^2\left(3-4\Omega\right)-A^2\left(5-8\Omega\right)\right], \qquad \text{and} \tag{121}$$

$$\beta(r,u) = \frac{1}{2}\ln\left[\mathrm{C}\left(u\right)\right]-\frac{1}{2}\ln\left[1-\frac{2m(r,u)}{r}\right]. \tag{122}$$

Next section is devoted to studied the physical contribution of the *NLES* to several collapsing models emerging from the above effective variables (equations (119) and (120)) and from the corresponding metric functions (121) and (122).

G. The *Systems of Surface Equations*

At this point two family of models will be considered. The first family emerges with $\mathrm{C} = const.$ which, by equation (107), clearly represents a configuration with a constant gravitational potential at the surface, i.e.

$$h_a = 1-2\frac{M}{A} = F = \mathrm{C} = const.. \tag{123}$$

The second family of models is the most general one. It is assumed that C is a function of the time like coordinate, i.e. $\mathrm{C} = \mathrm{C}(u)$.

In order to understand the contribution of the *NLES* to the hydrodynamic scenario during the collapse, we shall compare our models with *NLES* to the most similar standard Gokhroo-Mehra-Martínez model [20].

In the first case, $\mathrm{C} = const$, the *system of surface equations* is

$$\dot{A} = \mathrm{C}\left(\Omega-1\right), \tag{124}$$

$$L = -\frac{1}{2}\mathrm{C}\left(\Omega-1\right)\left(1-\mathrm{C}\right), \qquad \text{and} \tag{125}$$

$$\dot{\Omega} = \frac{2\,\Omega-1}{A\left(1-\mathrm{C}\right)}\left[L-\dot{A}\left(1-\mathrm{C}\right)\right]-\frac{\Omega-1}{2A}\left[1-11\,\mathrm{C}-4\,\mathrm{C}\left(8\Omega-9\right)\Omega\right]. \tag{126}$$

This system of nonlinear ordinary differential equations can be solved (numerically) for a given set of initial values for $A\left(u\right)$, $F\left(u\right)$ and $\Omega\left(u\right)$.

As we have stressed above, the reference model will be the Gokhroo-Mehra-Martínez model [20] with a constant gravitational potential at the surface of the configuration. In this model the only different equation is the third surface equation. Thus, the *system of surface equations* is conformed by equations (124), (125) and by the equation (4.28) in reference [20] assuming $\dot{F} = 0$. Notice that, in this case, the luminosity profile,$L\left(u\right)$, emerges as a consequence of the condition that the gravitational potential at the surface of the configuration be constant.

In the case $\mathrm{C} = \mathrm{C}(u)$, the *system of surface equations* is given by

$$\dot{A} = \mathrm{C}(u)\left[\Omega-1\right], \tag{127}$$

$$\dot{\mathrm{C}}(u) = \frac{2L+\mathrm{C}(u)\left[1-\mathrm{C}(u)\right]\left[\Omega-1\right]}{A}, \qquad \text{and} \tag{128}$$

$$\dot{\Omega} = \frac{2\,\Omega-1}{A\left(1-\mathrm{C}(u)\right)}\left[L-\dot{A}\left(1-\mathrm{C}(u)\right)-\frac{A\,\dot{\mathrm{C}}(u)}{2\,\mathrm{C}(u)}\right]-\frac{\Omega-1}{2A}\left[1-11\,\mathrm{C}(u)-4\,\mathrm{C}(u)\left(8\Omega-9\right)\Omega\right]. \tag{129}$$

The set of initial condition and parameters for three models are

$M(0) = 1.0M_{\odot}$	$\rho_c(0) = 2.2\times10^{14}\ g\ cm^{-3}$
$A(0) = 10$	$\rho_a(0) = 9.8\times10^{13}\ g\ cm^{-3}$
$\Omega(0) = 1-\epsilon$	$a(0) = 1477\,0\ m$
$K(0) = 5/9$	$z_a(0) = 0.118$
$\mathrm{C}(0) = 4/5$	$\omega_a\left(0\right) = -1.0\times10^{-10}\,c$

Concerning the above set, it should be pointed out that:

- All of them correspond to typical values for young neutron stars.

- Because of (90), $\Omega = 1$ corresponds to $\omega_a = 0$. From the above *systems of surface equations,* it is straightforward to verify that this initial velocity lead to static models.
- In order to obtain collapsing configurations the initial surface velocity has been perturbed with an $\epsilon \approx 10^{-10}$.
- In our simulations, we have imposed that the energy conditions for imperfect fluid be satisfied. In addition, the restrictions $-1 < \omega < 1$ and $r > 2m\,(u, r)$ at any shell within the matter configuration are also fulfilled.
- For C = C(u) :
  - we do not need any perturbation in the velocity to start the collapse.
  - The luminosity profile, $L\,(u)$, has been provided as a Gaussian pulse centered at $u = u_p$

$$-\dot{M} = L = \frac{\Delta M_{rad}}{\lambda\sqrt{2\pi}} \exp \frac{1}{2}\left(\frac{u - u_p}{\lambda}\right)^2, \tag{130}$$

with λ the width of the pulse and ΔM_{rad} the total mass lost in the process. This family of models has been simulated using

$$\Delta M_{rad} = 10^{-6}\, M(0), \qquad \lambda = 5, \qquad u_p = 100 \text{ ms}. \tag{131}$$

H. The Evolution of the physical variables

The evolution of the physical variables are presented as functions of the standard *Schwarzschild time.* The relationship between the usual Schwarzschild coordinates, (T, R, Θ, Φ), and the previously mentioned Bondi's radiation coordinates can be expressed as

$$u = T - \int \frac{r}{e^{2\beta}(r - 2m(u, r))}\, dr, \qquad \theta = \Theta, \qquad r = R \quad \text{and} \quad \phi = \Phi\ . \tag{132}$$

Figure 2 displays the evolution of the boundary of the surface for the Gokhroo-Mehra model with a *NLES* (plate (a)) and with constant gravitational potential at the surface (plate (b)). None of them tend to any asymptotic equilibrium configuration as was reported earlier [20]. It is apparent from these figures that our models with a *NLES* lead to a softer configurations than those without this particular equation of state. Our matter configuration starts to collapse earlier and evolves deeper than the reference Gokhroo-Mehra-Martínez model with $\dot{F} = 0$. In addition, because of (123) $M \propto A$, the models with a *NLES* radiates more energy than those without it.

The profiles of the physical variables (ρ, P_r, q , and ω/c) for three different times, are sketched in Figures 3 and 4 (plates (a) through (d)). Figure 3 displays the profiles for the Gokhroo-Mehra model with a *NLES* and Figure 4 Gokhroo-Mehra-Martínez model with constant gravitational potential at the surface. It can be appreciated from these figures that the changes in time of the density and pressure (Figure 3, plates (a) and (b)) of the configuration with the *NLES* are more significant than those observed in Figure 4, (plates (a) and (b)). In fact, from Figure 3 (b) it is clear that the pressure rises at the inner core while it diminishes at the outer mantle. This can be understood if we recall that the total radial pressure P_r collects the contribution of both the hydrodynamic and radiation pressures and that it is at the outer layers where the radiation pressure becomes more significant in time. Now, concerning plates (b) and (c), we can see that around the zone of $r = 5.5$ Kms., the total pressure remains constant and coincides with the maximum of the energy flux density. It is clear that the energy flux density in models with a *NLES* (Figure 3, plate (c)) is more intense than that observed in the Gokhroo-Mehra-Martínez reference model (Figure 4, plate (c)). Finally notice that in Figures 4(c) and 4(d), those shells that have a strong energy flux density also have the higher speed. This situation contrasts with the difference that can be observed in the Figure 3, (plates (c) and (d)) between the maximum of the energy flux density (around $r = 5.5$ Kms.) and the maximum in the velocity at the $r = 9.2$ Kms. This difference of about 4 Kms. remains constant in time during the collapse. This effect also appears in the *NLES* with variable gravitational potential at the surface (Figure 5, plates (c) through (d)), and is not present in the corresponding Gokhroo-Mehra-Martínez reported in [20]. It is worth mentioning that, because of our assumption of the diffusion regime approximation for the radiation flux, we expect that hydrodynamics and radiation should be closely related and this is clear in the Gokhroo-Mehra-Martínez models. Therefore, it seems that the shift between the radiation maxima and the velocities could be thought of as the main effect emerging from the integral contribution presented in the nonlocal equations of state (5) and (6).

Finally, in plates (e) and (f) in Figure 4 particular hydrodynamic discontinuities, that should be mentioned, are displayed. These plates zoom in time of the corresponding plates (c) and (d) of the same figure. As it can be appreciated from these plates, as the energy flux rises there is a wave front propagating outward. The discontinuity is present not in the physical variable (either the energy flux or the matter velocity) but in its derivative. This wave front, driven by the radiation pressure, can be associated to a hypersurface traveling with the velocity of sound [26]. It is clear that the shell where the discontinuity of the derivative of the energy flux takes place coincides with the corresponding surface where the discontinuity in the derivative of the velocity is identified and this coincidence is consistent with the radiation regime that we have assumed.

VI. CONCLUDING REMARKS

We have shown that fluids satisfying nonlocal equations of state (either (5) or (6)) could represent feasible bounded matter configurations and some of them could have significant geometric properties.

Some answers to the questions that motivate the present work have been explored:

- Assuming (26) and C as a constant parameter in (2), we have found that spherically symmetric space-times described by (32), which represent fluids satisfying a *NLES,* have a *CKV* of the form (30).
- The anisotropic imperfect fluid, emerging from the metric (32), fulfills the corresponding energy conditions.
- Bounded gravitational sources can materialize from this type of fluids and meaningful hydrodynamic scenarios can be obtained.
- Although the simulations for collapsing configuration where carried out for a very short period of time, it seems that the radiating spheres we have considered could represent some initial phases in the evolution of compact objects.

More investigations on this curious equation of state are needed to extend our understanding of the possible collective effects that are clear in either (5) or (6). But, as the main conclusion of the present work it can be asserted that *Nonlocal Equations of State* could represent satisfactory candidates to describe the behavior of matter at supranuclear densities in General Relativity.

VII. ACKNOWLEDGMENTS

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[1] Demiański M., 1985, *Relativistic Astrophysics,* in International Series in Natural Philosophy, Vol 110, Edited by *D. Ter Haar*, (Pergamon Press, Oxford).
[2] Shapiro S.L., Teukolsky S.A., 1983, *Black Holes, White Dwarfs and Neutron Stars*, (John Willey, New York).
[3] Narasimhan, M.N.L., 1993, *Principles of Continuum Mechanics*, (John Willey, New York) p. 510.
[4] Mihalas, D. and Mihalas, B. 1984, *Foundations of Radiation Hydrodynamics* (Oxford University Press, Oxford).
[5] Hernández H. and Núñez L.A., 1998, "Anisotropic Spheres with Non-Local Equation of State in General Relativity" in preparation.
[6] Lindblom L. 1993. in **Classical General Relativity.** Edited by *S. Chandrasekhar*. (Oxford University Press England).
[7] Herrera L., Jiménez J., Leal L., Ponce de León J., Esculpi M., and Galina V., 1984, *J. Math. Phys.*, **25**, 3274.
[8] Duggal, K.L. and Sharma, J., 1986, *J. Math. Phys.*, **27**, 2511.
[9] Maartens, R., Mason and Tsamparlis, D., 1986, *J. Math. Phys.*, **28**, 1114.
[10] Duggal, K.L., 1987, *J. Math. Phys.*, **28** , 2700.
[11] Herrera L and Santos N.O., 1997, *Physics Reports,* **286**, 53.
[12] Kolassis C. A., Santos N.O., and Tsoubelis D., 1988, *Class. Quantum Grav.*, **5**, 1329.
[13] Bondi, H., 1964 *Proc. Roy. Soc. London,* **281,** 39.
[14] Bonnor W.B., De Oliveira A.K.G., Santos N.O., 1989, *Physics Reports*, **181**, 269.
[15] Herrera L., 1996, in **Notas de la II Escuela Venezolana de Relatividad, Campos y Astrofísica**, Edited by *H. Rago*, (Universidad de los Andes, Mérida), p. 81.
[16] Kippenhahn, R. and Weigert, A., 1990, *Stellar Structure and Evolution*, (Springer Verlag, Berlin), p. 11.
[17] Herrera, L., Jiménez, J. and Ruggeri, G. 1980, *Phys. Rev.* **D22** 2305.
[18] Herrera, L. and Núñez, L. A., 1990, *Fund. Cosmic Phys.* **14**, 235.
[19] Bondi, H., Van der Burg, M. G. J. and Metzner, A. W. K., 1962, *Proc R. Soc. London,* **A269**, 21.
[20] Martínez J., 1996, *Phys. Rev. D*, **53** , 6921.
[21] Aguirre F., Hernández H., and Núñez L.A., 1994, *Astrophys. Space Sci.*, **219,** 153.
[22] Cosenza M., Herrera L., Esculpi M., and Witten L., 1982, *Phys. Rev. D*, **25**, 2527.
[23] Gokhroo M.K., Mehra A.L., 1994, *Gen. Rel. Grav.*, **26**, 75.
[24] Herrera, L. and Martínez J., 1998a, *Gen. Rel. Grav.*, **30**, 445.
[25] Herrera, L. and Martínez J., 1998b, *Class. Quantum Grav.*, **15,** 404.
[26] Taub A. H., 1983, in **Proceeding of the Third Marcel Grossmann Meeting on General Relativity** Edited by *H. Ning*, (Science Press and North Holland Pub. Co. Amsterdam) p.165.

Figure Captions

Figure 1 The total evolution times Tev_I and Tev_{II} are sketched as function of the initial mass. In both cases we have assumed 10 Kms. as the value for the initial radius of the collapsing configurations.

Figure 2 Evolution of the boundary radius of the matter configuration satisfying *NLES*, (plate (a)) and for the standard Gokhroo and Mehra model with constant gravitational potential at the surface constant gravitational potential at the surface (plate (b)).

Figure 3 Evolution of matter variables for the Gokhroo and Mehra model with a *NLES*. Plates (a) through (d) represent the profiles for the density, pressure, energy flux density and mass velocity, respectively at three different times.

Figure 4 Evolution of matter variables for the standard Gokhroo and Mehra model with constant gravitational potential at the surface. Plates (a) through (d) represent the profiles for the density, pressure, energy flux density and mass velocity, respectively at three different times. Plates (e) and (f) are a zoom in time to the corresponding plates (c) and (d), respectively.

Figure 5 Evolution of matter variables satisfying *NLES* with $F(u) = 1 - \frac{2M(u)}{A(u)} = \mathrm{C}(u)$. Plates (a) through (d) represent the profiles for the density, pressure, energy flux density and mass velocity, respectively at three different times.

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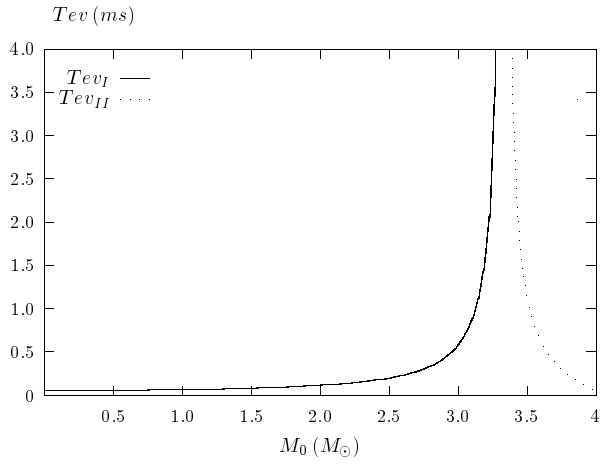


FIGURE 1

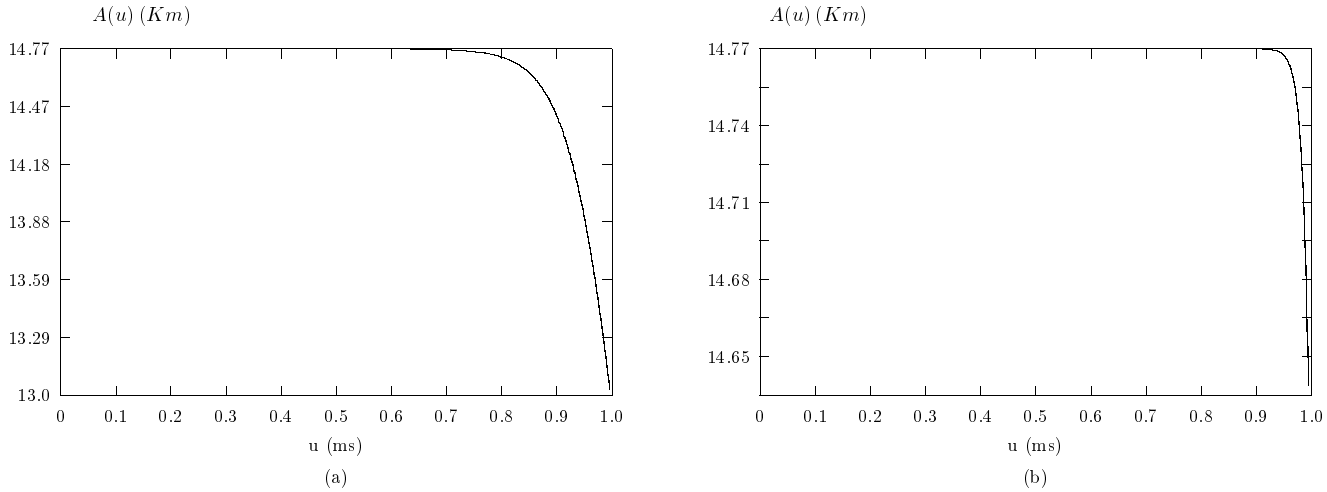
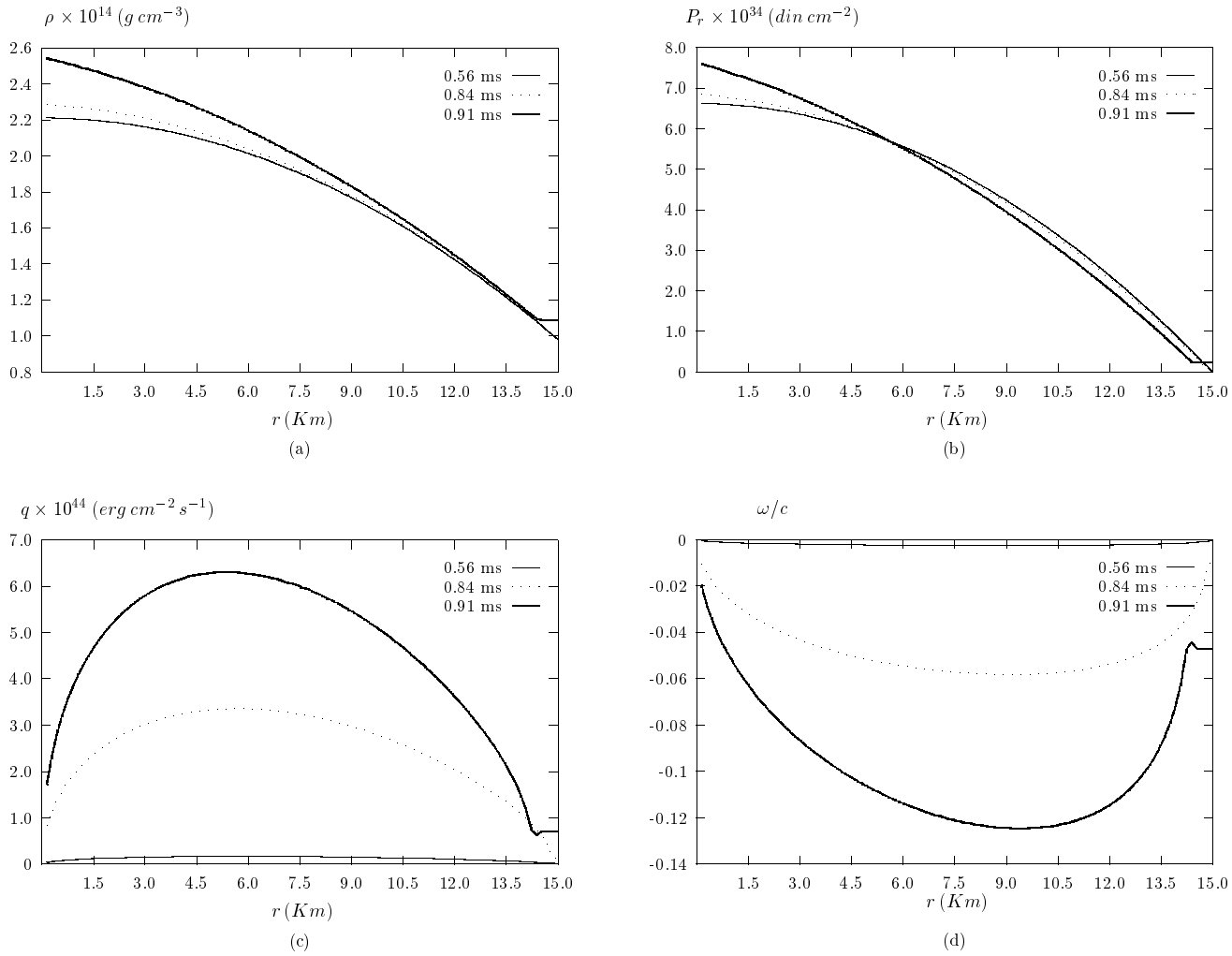


FIGURE 2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0.56 ms
0.84 ms
0.91 ms
$r\,(Km)$
(a)
$P_r \times 10^{34}\,(din\,cm^{-2})$
0.56 ms
0.84 ms
0.91 ms
$r\,(Km)$
(b)
$q \times 10^{44}\,(erg\,cm^{-2}\,s^{-1})$
0.56 ms
0.84 ms
0.91 ms
$r\,(Km)$
(c)
ω/c
0.56 ms
0.84 ms
0.91 ms
$r\,(Km)$
(d)

FIGURE 3

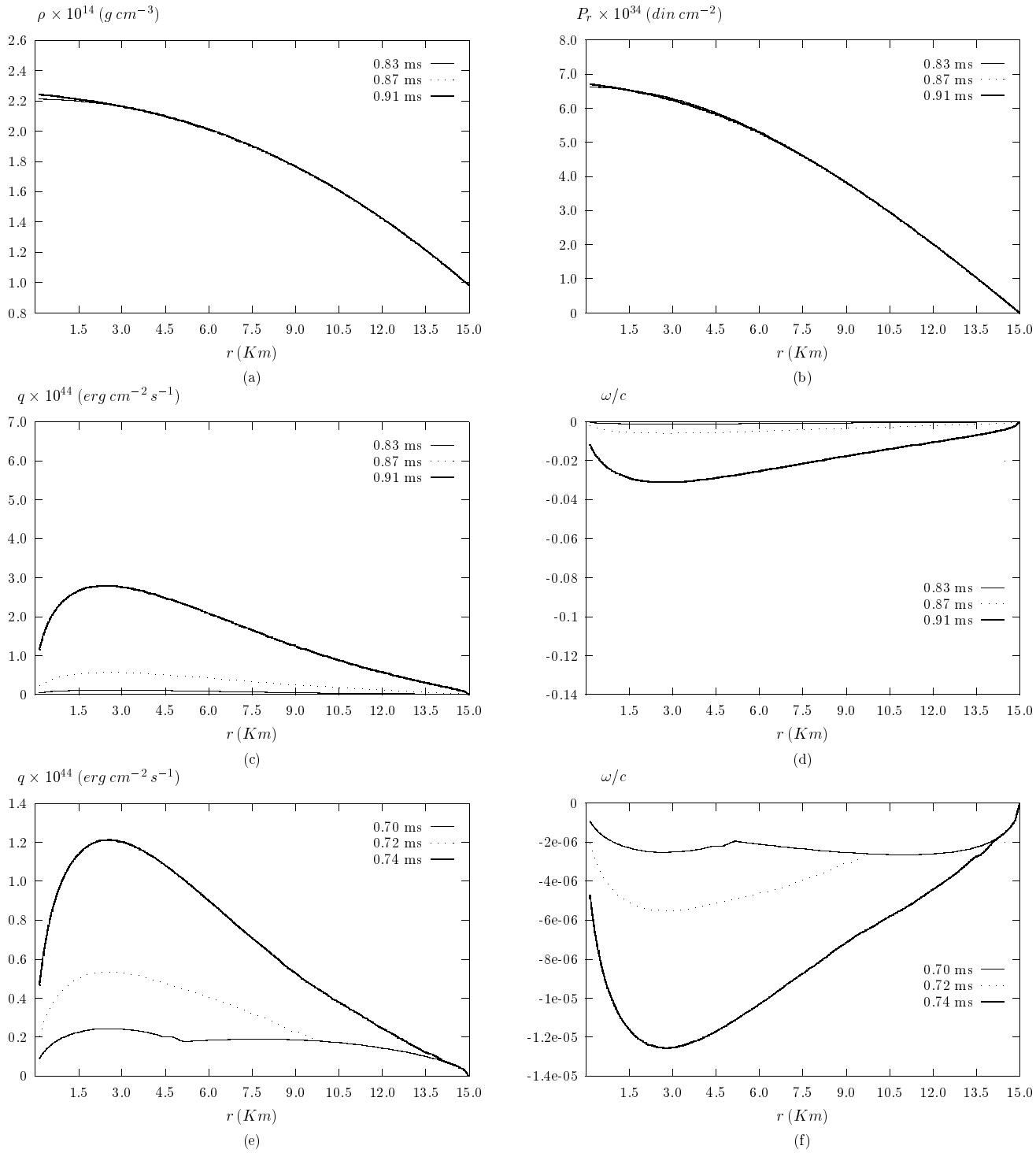
$\rho \times 10^{14}\,(g\,cm^{-3})$
0.83 ms
0.87 ms
0.91 ms
$r\,(Km)$
(a)
$P_r \times 10^{34}\,(din\,cm^{-2})$
0.83 ms
0.87 ms
0.91 ms
$r\,(Km)$
(b)
$q \times 10^{44}\,(erg\,cm^{-2}\,s^{-1})$
0.83 ms
0.87 ms
0.91 ms
$r\,(Km)$
(c)
ω/c
0.83 ms
0.87 ms
0.91 ms
$r\,(Km)$
(d)
$q \times 10^{44}\,(erg\,cm^{-2}\,s^{-1})$
0.70 ms
0.72 ms
0.74 ms
$r\,(Km)$
(e)
ω/c
0.70 ms
0.72 ms
0.74 ms
$r\,(Km)$
(f)

FIGURE 4

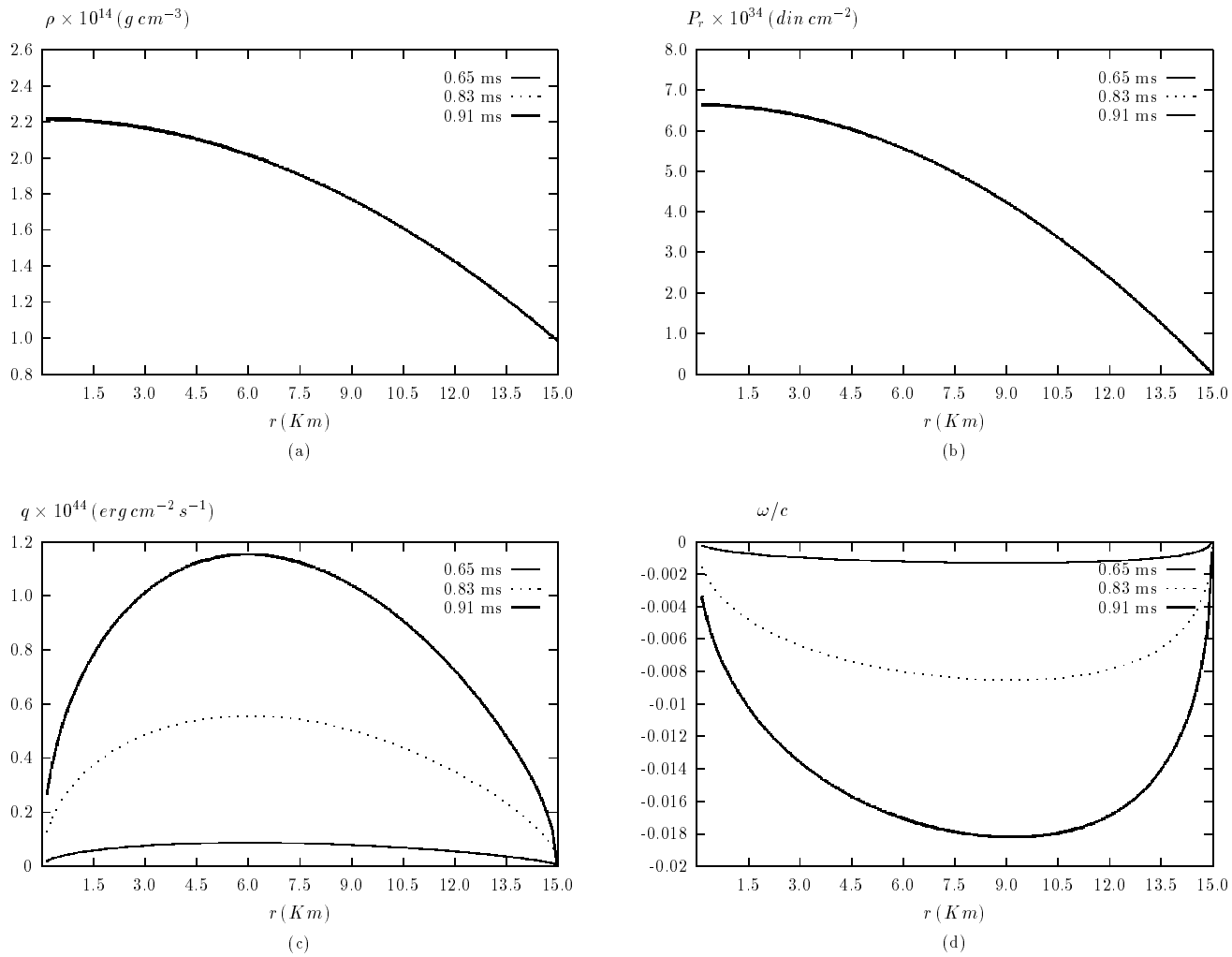
$\rho \times 10^{14}\,(g\,cm^{-3})$
0.65 ms
0.83 ms
0.91 ms
$r\,(Km)$
(a)
$P_r \times 10^{34}\,(din\,cm^{-2})$
0.65 ms
0.83 ms
0.91 ms
$r\,(Km)$
(b)
$q \times 10^{44}\,(erg\,cm^{-2}\,s^{-1})$
0.65 ms
0.83 ms
0.91 ms
$r\,(Km)$
(c)
ω/c
0.65 ms
0.83 ms
0.91 ms
$r\,(Km)$
(d)

FIGURE 5