Homothetic perfect fluid spacetimes

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Abstract. A brief summary of results on homotheties in general relativity is given, including general information about spacetimes admitting an r-parameter group of homothetic transformations for $r > 2$, as well as some specific results on perfect fluids. Attention is then focused on inhomogeneous models, in particular on those with a homothetic group H_4 (acting multiply transitively) and H_3. A classification of all possible Lie algebra structures along with (local) coordinate expressions for the metric and homothetic vectors is then provided (irrespectively of the matter content) and some new perfect fluid solutions are given and briefly discussed.

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1. Introduction

This paper is devoted to the study of spacetimes admitting an intransitive group of homotheties, with a view towards those which can be interpreted as perfect fluid solutions of Einstein's field equations [1].

A collection of important results regarding generic properties of spacetimes admitting homothetic transformations can be found in [2–7] (and references cited therein), and in [8] where the case of multiply transitive action is thoroughly studied by Hall and Steele.

The study of this subject began with the pioneering paper by Cahill and Taub [9], followed by the works of Eardley [10, 11]. From then on, homotheties have been studied in connection with a wealth of situations of physical interest in classical general relativity as well as in cosmology, see [12–14] for interesting reviews on homothetic solutions.

The paper is organized as follows. Section 2 contains a brief summary of results on groups of homotheties and spacetimes admitting them, the implications that they have on perfect fluids and the physical quantities characterizing them (density, pressure, velocity, etc), and we also summarize all the general information about spacetimes admitting an r-parameter group of homothetic transformations for $r > 2$. This includes the dimension of the homothetic and isometric algebras as well as that of the orbits they act on respectively, together with its nature (spacelike, timelike or null), allowed Petrov and Segre types of Weyl and Ricci tensors, and whether perfect fluid solutions exist or not. Most of the contents of this section is a review of diverse results in the literature, which we considered useful to gather together in a single table. Some of the results are, as far as we are aware of, new (see especially those cases where null orbits occur).

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In sections 3 and 4 a classification of all possible Lie algebra structures along with (local) coordinate expressions for the metric and homothetic vectors, are given for the cases H_4 (acting multiply transitively on three-dimensional orbits, in keeping with our assumption of intransitive action) and H_3, respectively. The characterizations provided are independent of the field equations, and therefore they may have applications other than those considered here (perfect fluids).

In particular, section 3 contains some review material on the case $r = 4$, together with some new results, all of them presented in a unified manner, extending the work of Wu [15], Cahill and Taub [9] and Shikin [17]. The general perfect fluid solution is then given in certain, well defined and invariantly characterized subcases. Whenever this is not possible, a few selected examples are presented. It is assumed that the matter satisfies the weak and dominant energy conditions and expressions for the kinematical quantities (acceleration, expansion, deceleration parameter, shear and vorticity) are provided for each case.

Finally, section 4 contains some new solutions for the case $r = 3$ and appropriate references to related work on this issue. We summarize the results concerning the topology of the Killing orbits and the Bianchi classification of the homothetic algebras. We distinguish the cases where the Killing subalgebra is Abelian from that where it is non-Abelian. Attention is then restricted to the orthogonally transitive case, giving for each possible Lie algebra structure the coordinate forms of the proper homothetic vector field and the metric. In the Abelian case we distinguish three different classes of such models, depending on the orientation of the fluid flow relative to the homothetic orbits. The case in which we are more interested is the so-called 'tilted' case where new solutions are found. These models play a significant role in theoretical cosmology, since they can represent [44] asymptotic states of more general G_2 cosmologies (see section 4.1). Finally, we provide explicit forms for the homothetic vector field and the metric in the case of a (maximal) non-Abelian G_2, and, although no perfect fluid solutions have been found, we briefly discuss some properties.

2. Basic facts about homotheties

2.1. Definition and properties

Throughout this paper (M, g) will denote a spacetime: M is a Hausdorff, simply connected, four-dimensional manifold and g a Lorentz metric of signature $(-, +, +, +)$. All the structures will be assumed to be smooth.

A global vector field X on M is called homothetic if either one of the following equivalent conditions holds on a local chart:

$$\mathcal{L}_X g_{ab} = 2n g_{ab}, \qquad X_{a;b} = n g_{ab} + F_{ab}, \tag{1}$$

where n is a constant on M, $\mathcal{L}$ stands for the Lie derivative operator, a semicolon denotes a covariant derivative with respect to the metric connection and $F_{ab} = -F_{ba}$ is the so-called homothetic bivector. If $n \neq 0$, X is called proper homothetic and if $n = 0$, X is a Killing vector (KV) on M. For a geometrical interpretation of (1) we refer the reader to [2, 6].

A necessary condition for X to be homothetic is

$$X^a{}_{;bc} = R^a{}_{bcd} X^d, \tag{2}$$

where $R^a{}_{bcd}$ are the components of the Riemann tensor in the above chart; thus, a homothetic vector field (HVF) is a particular case of affine collineation [7] and therefore it will satisfy

$$\mathcal{L}_X R^a{}_{bcd} = \mathcal{L}_X R_{ab} = \mathcal{L}_X C^a{}_{bcd} = 0, \tag{3}$$

where R_{ab} ($\equiv R^c{}_{acb}$) and $C^a{}_{bcd}$ stand, respectively, for the components of the Ricci and the conformal Weyl tensor.

The set of all HVFs on M forms a finite-dimensional Lie algebra under the usual bracket operation and will be referred to as the homothetic algebra, $\mathcal{H}_r$, where r is its dimension. The set of all KVs on M also forms a finite-dimensional Lie algebra, the Lie algebra of isometries, which will be denoted by $\mathcal{G}_s$ (where s is its dimension), and one has that $\mathcal{G}_s \subseteq \mathcal{H}_r$ (i.e. $\mathcal{G}_s$ is a subalgebra of $\mathcal{H}_r$). Furthermore, it can be seen by direct computation that the Lie bracket of an HVF with a KV is always a KV and that, given any two proper HVFs, there always exists a linear combination of them which is a KV. From these considerations it immediately follows that the highest possible dimension of $\mathcal{H}_r$ in a four-dimensional manifold is $r = 11$.

If $r \neq s$ then $s = r - 1$ necessarily, and one may choose a basis $\{X_1, \ldots, X_{r-1}, X\} \equiv \{X_A\}_{A=1...r}$ for $\mathcal{H}_r$, in such a way that X is proper homothetic and $X_1, \ldots, X_{r-1}$ are Killing vector fields spanning $\mathcal{G}_{r-1}$. If these vector fields in the basis of $\mathcal{H}_r$ are all complete vector fields, then each member of $\mathcal{H}_r$ is complete and Palais' theorem [4, 18, 19] guarantees the existence of an r-dimensional Lie group of homothetic transformations of M (H_r) in a well known way; otherwise, it gives rise to a local group of local homothetic transformations of M and, although the usual concepts of isotropy and orbits still hold, a little more care is required [8].

The following result [8, 20] will be useful:

The orbits associated with $\mathcal{H}_r$ and $\mathcal{G}_{r-1}$ can only coincide if either they are four dimensional or three dimensional and null. (This result still holds if $\mathcal{H}_r$ is replaced by the conformal Lie algebra $\mathcal{C}_r$ and does not depend on the maximality of $\mathcal{H}_r$ or $\mathcal{C}_r$).

The set of zeros of a proper HVF, i.e. $\{p \in M : X(p) = 0\}$ (fixed points of the homothety), either consists of topologically isolated points or else is part of a null geodesic. The latter case corresponds to the well known (conformally flat or Petrov type N) plane waves [2, 21].

At any zero of a proper HVF on M all Ricci and Weyl eigenvalues must necessarily vanish and thus the Ricci tensor is either zero or has Segre type $\{(2, 11)\}$ or $\{(3, 1)\}$ (both with zero eigenvalue), whereas the Weyl tensor is of the Petrov type O, N or III [2] (see also [22] for vacuum spacetimes).

2.2. Perfect fluids

The energy–momentum tensor of a perfect fluid is given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}, \tag{4}$$

where μ and p are, respectively, the energy density and the pressure as measured by an observer comoving with the fluid, and u^a ($u^a u_a = -1$) is the 4-velocity of the fluid. If X is an HVF then, from Einstein's field equations (EFE) it follows that

$$\mathcal{L}_X T_{ab} = 0, \tag{5}$$

and this implies in turn [10]

$$\mathcal{L}_X u_a = n u_a, \qquad \mathcal{L}_X p = -2np, \qquad \mathcal{L}_X \mu = -2n\mu. \tag{6}$$

Thus, the Lie derivatives of u_a, p and μ with respect to a KV vanish identically.

If a barotropic equation of state exists, $p = p(\mu)$, and the spacetime admits a proper HVF X then [23]

$$p = (\gamma - 1)\mu, \tag{7}$$

where γ is a constant ($0 \leqslant \gamma \leqslant 2$ in order to comply with the weak and dominant energy conditions). Of particular interest are the values $\gamma = 1$ (pressure-free matter, 'dust') and $\gamma = \frac{4}{3}$ (radiation fluid). In addition, the value $\gamma = 2$ (stiff matter) has been considered in connection with the early Universe. Furthermore, values of γ satisfying $0 \leqslant \gamma < \frac{2}{3}$, while physically unrealistic as regards a classical fluid, are of interest in connection with inflationary models of the Universe. In particular, the value $\gamma = 0$, for which the fluid can be interpreted as a positive cosmological constant, corresponds to exponential inflation, while the values $0 < \gamma < \frac{2}{3}$ correspond to power-law inflation in FRW models [24], but it is customary to restrict γ to the range $1 \leqslant \gamma \leqslant 2$.

If the proper HVF X and the 4-velocity u are mutually orthogonal (i.e. $u^a X_a = 0$) and a barotropic equation of state is assumed, it follows that $\gamma = 2$, i.e. $p = \mu$ stiff-matter [10], on the other hand, if $X^a = \alpha u^a$ the fluid is then shear-free. Further information on this topic can be found in [25–27].

2.3. The 'dimensional countdown'

In this subsection, the maximal Lie algebra of global HVF on M will be denoted as $\mathcal{H}_r$ (r being its dimension) and it will be assumed that at least one member of it is proper homothetic.

The case of multiply transitive action is thoroughly studied in [8], and we shall refer the reader there for details; nevertheless, and for the sake of completeness, we summarize in table 1 the results given there, which follow invariably from considerations on the associated Killing subalgebra and the fixed-point structure of the proper HVF. Furthermore, we have

Table 1.

r	O_m	K_n	Petrov	Segre	Interpretation	PF, info.
11	M	M	O	0	Flat	$\nexists$
10	M	M	—	—	Not possible	$\nexists$
9	M	M	—	—	Not possible	$\nexists$
8	M	M	O	$\{(2, 11)\}$	Gen. plane wave	$\nexists$
7	M	M	N	$0, \{(2, 11)\}$	Gen. plane wave	$\nexists$
7	M	T_3	O	$\{(1, 11)1\}$	Tachyonic fluid	$\nexists$
7	M	N_3	—	—	Not possible	$\nexists$
7	N_3	N_3	O	$\{(2, 11)\}$	Gen. plane wave	$\nexists$
7	M	S_3	O	$\{1, (111)\}$	Perfect fluid	FRW
6	M	M	—	—	Not possible	$\nexists$
6	N_3	N_3	N	$\{(2, 11)\}$	Gen. plane wave	$\nexists$
5	M	M	—	—	Not possible	$\nexists$
5	M	N_3	—	—	—	$\nexists$
5	N_3	N_3	—	—	Not possible	$\nexists$
5	M	T_3	D, N, O	$\{1, 1(11)\}, \{2, (11)\}$		LRS
5	M	S_3	D, O	$\{(1, 1)11\}, \{(2, 1)1\}$		LRS
4	M	N_3	II, III, D, N, O	$\{(1, 1)(11)\}, \{(2, 11)\}$	Plane waves	$\nexists$
4	N_3	N_3	—	—	Not possible	$\nexists$
4	M	T_3				Bianchi
4	M	S_3				Bianchi
4	O_3	N_2	N, O	$\{3, 1\}, \{2, 11\}, \{(1, 1)11\}$		$\nexists$
4	O_3	T_2	D, O	$\{(1, 1)11\}$		$\nexists$
4	O_3	S_2	D, O	$\{-(11)\}$		$\exists$

added a few other results, also in the literature or following straightforwardly from those, so as to complete the study down to dimension 4.

The first entry in the table gives the dimension of the group of homotheties, the second and third entries stand for the nature and dimension of the homothetic and Killing orbits, respectively (e.g. N_2, T_2 and S_2 denote null, timelike and spacelike two-dimensional orbits, respectively, O_3 stands for three-dimensional orbits of either nature, timelike, spacelike or null), the fourth and fifth entries give the Petrov and Segre type(s) of the associated Weyl and Ricci tensors (in the latter case it is to be understood that all possible degeneracies of the given types, can in principle occur, including vacuum when possible). Finally, the last two entries give, respectively, the possible interpretation whenever it is in some sense unique, and the existence or non-existence of perfect fluid solutions for that particular case, along with some supplementary information; thus FRW stands for Friedmann–Robertson–Walker, LRS for locally rotationally symmetric and Bianchi refers to that family of perfect fluid solutions. The cases that cannot arise are labelled as 'Not possible', and wherever no information is given on the Petrov and Segre types, it is to be understood that all types are possible in principle. The Segre type of the Ricci tensor of the case described in the last row, is unrestricted except in that it must necessarily have two equal (spacelike) eigenvalues; perfect fluid solutions of these characteristics constitute special cases of spherically, plane or hyperbolically symmetric perfect fluid spacetimes. For further information on LRS spacetimes, see [28, 29]; for the case $r = 4$ transitive and null three-dimensional Killing orbits, see [1, 30]. Regarding spatially homogeneous Bianchi models, see [30, 31, 36, 37] and for the last three cases occurring in the table, see respectively [32, 33], [1, 33] and [1].

The case of $r = 3$ has an associated Killing subalgebra $\mathcal{G}_2$ and the respective dimensions of their orbits are 3 and 2 (see for instance [34, 35, 40, 43] and references cited therein). When the Killing subalgebra has null orbits, the metric is of Kundt's class [50] and perfect fluids are excluded. If the Killing orbits are timelike, the solutions can then be interpreted as special cases of axisymmetric stationary spacetimes (provided that regularity conditions hold on the axis [1, 40]), and if they are spacelike as special cases of inhomogeneous cosmological solutions or cylindrically symmetric spacetimes. In both cases, perfect fluid solutions have been found.

3. The H_4 case

The associated isometric group for perfect fluid spacetimes acts necessarily on T_3, S_3 or S_2 orbits (see table 1). In the intransitive case, the G_3 must act multiply transitively on two-dimensional surfaces of maximal symmetry S_2, which are then of constant (positive, zero or negative) curvature and admit orthogonal surfaces [45].

Possibly no problem in this context has been more exhaustively studied than that of spherically symmetric homothetic spacetimes, beginning with the seminal paper of Cahill and Taub [9] and continuing with recent papers by Ori and Piran [46], Carr and Yahil [47], Henriksen and Patel [48], and Foglizzo and Henriksen [49] among others (see references therein). Homothetic spacetimes with plane symmetry are also considered by Shikin [17].

What we attempt in this section, rather than presenting a survey of the models existing in the literature, is to study in a unified manner all possible cases, i.e. homothetic orbits of either nature (timelike, spacelike or null at every point $p \in M$ and the more general case in which their nature varies from point to point) as well as the different possibilities for the curvature k of the isometry orbits (spherical and plane symmetry as well as the $k = -1$ case); thus, extending previous works by Wu [15] and Nilsson and Uggla [16], where only spacelike homothetic orbits are considered (i.e. type B and some type C solutions in our

classification below), and by Cahill and Taub [9] and Shikin [17] where only spherical and plane symmetry are considered, respectively. We carefully examine all possible Lie algebras and related metrics. In so doing we are able to obtain some new perfect fluid solutions for the type C, as well as to exhaust the study for types A and B rediscovering some known solutions.

As is well known, the spacetime metric can be written as [1]

$$ds^2 = -A^2(r, t)\, dt^2 + B^2(r, t)\, dr^2 + F^2(r, t)(dy^2 + \Lambda^2(y, k)\, dz^2), \tag{8}$$

$$\Lambda(y, k) = \begin{cases} \sin y & k = +1 \\ y & k = 0 \\ \sinh y & k = -1. \end{cases} \tag{9}$$

The Killing vectors are $\xi_1 = \sin z\, \partial_y + (\Lambda'/\Lambda) \cos z\, \partial_z$, $\xi_2 = \cos z\, \partial_y - (\Lambda'/\Lambda) \sin z\, \partial_z$ and $\xi_3 = \partial_z$, where the prime denotes a derivative with respect to y and they satisfy the following commutation relations:

$$[\xi_1, \xi_2] = k\xi_3, \qquad [\xi_2, \xi_3] = \xi_1, \qquad [\xi_3, \xi_1] = \xi_2. \tag{10}$$

Assuming the existence of a proper HVF, X, and since its commutator with a KV must be a KV, the Jacobi identities imply the following structures for $\mathcal{H}_4$:

(I) $[X, \xi_i] = 0,$ $i = 1, 2, 3,$ $k = 0, \pm 1,$ (11)

$$X = X^t(t, r)\partial_t + X^r(t, r)\partial_r, \tag{12}$$

(II) $[X, \xi_1] = \xi_1,$ $[X, \xi_2] = \xi_2,$ $[X, \xi_3] = 0,$ $k = 0,$ (13)

$$X = X^t(t, r)\partial_t + X^r(t, r)\partial_r - y\partial_y. \tag{14}$$

Requiring the coordinate system to be a comoving one (i.e. $u_a = -A(t, r)\delta_a^t$) the HVF takes the form

$$X = X^t(t)\partial_t + X^r(r)\partial_r + X^y(y)\partial_y \equiv \hat{X} + X^y(y)\partial_y, \tag{15}$$

and the following possibilities then arise:

(A) $\hat{X} = \partial_t,$ (16)

(B) $\hat{X} = \partial_r,$ (17)

(C) $\hat{X} = \partial_t + \partial_r.$ (18)

The form (A) corresponds to u being tangent to the timelike homothetic orbits, $\mathcal{T}_3$, (B) corresponds to the case of spacelike homothetic orbits, $\mathcal{S}_3$, orthogonal to the fluid flow and (C) is the most general (tilted) case, also including the possibility of having null homothetic orbits, $\mathcal{N}_3$, when the functions $A(t, r)$ and $B(t, r)$ in (8) equal each other. The tilted case, assuming spatial self-similarity, has been investigated by Nilsson and Uggla [16] by using the qualitative theory of plane autonomous systems.

The homothetic equation (1) specialized to the metric (8) yields then the following possibilities:

Table 2. Case (A).

A	k	X	$A^2(t, r)$	$B^2(t, r)$	$F^2(t, r)$
I	$-1, 0, 1$	∂_t	$e^{2t} H^2(r)$	$e^{2t} H^2(r)$	$e^{2t} f^2(r)$
II	0	$\partial_t - y\partial_y$	$e^{2nt} H^2(r)$	$e^{2nt} H^2(r)$	$e^{2(n+1)t} f^2(r)$

where t has been scaled in AI so as to make $n = 1$ in (1). Now solving the field equations for a perfect fluid in each one of the above two cases, we find that the only possible solution for the family AI has an equation of state $\mu + 3p = 0$ and it admits (at least) one further Killing vector, thus being a particular case of $\mathcal{H}_5$ and we shall not study it here.

For AII, two families arise which depend on the value of n as shown in table 3

Table 3.

AII	$H(r)$	$f(r)$	μ	γ
$n \in (-\infty, -3) \cup$ $(-2, -1) \cup (0, +\infty)$	$a(\cosh(\alpha r))^{-1/\alpha}$	$(H/a)^{n+1}$	$(\beta/a^{2\alpha})e^{-2nt}H^{2(\alpha-1)}$	$2n/(2+3n)$
$n \in (-3, -2)$	$a(\sinh(\alpha r))^{-1/\alpha}$	$(H/a)^{n+1}$	$-(\beta/a^{2\alpha})e^{-2nt}H^{2(\alpha-1)}$	$2n/(2+3n)$

In table 3 α and β are given by

$$\alpha = \frac{2(n+1)}{n+2}, \qquad \beta = \frac{(n+1)(n+3)(3n+2)}{n^2(n+2)}, \tag{19}$$

and a is a constant. The vorticity is zero and the volume expansion θ, deceleration parameter $q \equiv -1 - 3\dot{\theta}/\theta^2$, acceleration $\dot{u}_a$ and non-vanishing shear tensor components σ_{ab} can be given as

$$\theta = \frac{2+3n}{H}e^{-nt}, \qquad q = \frac{-2}{2+3n}, \qquad \dot{u} = \frac{H'}{H}\partial_r, \tag{20}$$

$$\sigma_{rr} = -\frac{2e^{nt}H}{3}, \qquad \sigma_{yy} = \frac{e^{(2+n)t}f^2}{3H}, \qquad \sigma_{zz} = \frac{e^{(2+n)t}y^2f^2}{3H}. \tag{21}$$

Notice that, depending on the value of n, the solution contracts and decelerates or it expands and inflates.

With regard to the dimensionless scalars, the density parameter $\Omega \equiv 3\mu/\theta^2$, the dimensionless acceleration $W \equiv \dot{u}/\theta$ and the shear parameter $\Sigma \equiv 3\sigma^2/\theta^2$ (see [42] for further details), one has for the first family

$$\Omega = \frac{3(n+1)(n+3)}{(2+3n)n^2(n+2)}\frac{1}{\cosh^2(\alpha r)}, \qquad W = \frac{1}{2+3n}\tanh(\alpha r), \tag{22}$$

the models are then acceleration dominated at large distances, whereas for the second family

$$\Omega = -\frac{3(n+1)(n+3)}{(2+3n)n^2(n+2)}\frac{1}{\sinh^2(\alpha r)}, \qquad W = \frac{1}{2+3n}\coth(\alpha r), \tag{23}$$

thus being asymptotically spatially homogeneous. In both cases $\lim_{r\to\infty}\Omega = 0$ (vacuum-dominated models) and $\Sigma = 1/(2+3n)^2$, as this quantity is non-vanishing for all possible values of n, the models have no isotropic limit. Notice that the solution in the first case has been previously given in [42] and that in the second case the solution is singular at $r = 0$.

Table 4. Case (B). r in case BI has been re-scaled so as to have $n = 1$.

B	k	X	$A^2(t,r)$	$B^2(t,r)$	$F^2(t,r)$
I	$-1, 0, 1$	∂_r	$e^{2r}H^2(t)$	$e^{2r}H^2(t)$	$e^{2r}f^2(t)$
II	0	$\partial_r - y\partial_y$	$e^{2nr}H^2(t)$	$e^{2nr}H^2(t)$	$e^{2(n+1)r}f^2(t)$

Table 5.

BI	$H(t)$	$f^2(t)$	μ	Domain
$k = \pm 1$	1	$\mathrm{e}^{2t} + \frac{1}{2}k$	$\dfrac{k^2}{4\mathrm{e}^{2r}f^4}$	$t : \mathrm{e}^{2t} + k/2 > 0$
$k = 0, \pm 1$	1	$\alpha \sinh 2t + \frac{1}{2}k$	$\dfrac{k^2 + 4\alpha^2}{4\mathrm{e}^{2r}f^4}$	$t : \alpha \sinh 2t + k/2 > 0$
$k = \pm 1$	1	$\alpha \cosh 2t + \frac{1}{2}k$	$\dfrac{k^2 - 4\alpha^2}{4\mathrm{e}^{2r}f^4}$	$\alpha \in (-\frac{1}{2}, \frac{1}{2})$ $t : \alpha \cosh 2t + k/2 > 0$

Now solving the field equations for a perfect fluid source, one has table 5 where α is an arbitrary constant, which must be different from zero to prevent the occurrence of further Killing vectors. $k \neq 0$ in both the first and last cases since otherwise one would have a vacuum solution in the former case, and negative energy density in the latter. In all three cases, the fluid is irrotational and has an stiff equation of state, i.e. $\gamma = 2$. For these solutions one has

$$\theta = \frac{2\dot{f}}{\mathrm{e}^r f}, \qquad q = \frac{1}{2} - \frac{3}{2}\frac{f\ddot{f}}{(\dot{f})^2}, \qquad \dot{u} = -\partial_r, \qquad W = \frac{f}{2\dot{f}}, \qquad (24)$$

$$\sigma_{rr} = -\frac{2\mathrm{e}^r \dot{f}}{3f}, \qquad \sigma_{yy} = \frac{\mathrm{e}^r f \dot{f}}{3}, \qquad \sigma_{zz} = \frac{\mathrm{e}^r \Lambda^2 f \dot{f}}{3}, \qquad (25)$$

with a dot indicating a derivative with respect to t. In all cases the shear parameter is $\Sigma = \frac{1}{4}$, thus there is no isotropic limit. For all solutions in BI $\lim_{t \to \infty} W = \frac{1}{2}$, thus corresponding to accelerated dominated models, hence with no FRW limit. Regarding BII, we have the values given in table 6, with a, S and C given by, respectively,

$$a = \text{constant}, \qquad S = \sinh(2(n+1)t), \qquad C = \cosh(2(n+1)t). \quad (26)$$

Table 6.

BII	$H(t)$	$f^2(t)$	μ
$n < -3$ $n > -1$	$a[S]^{1/2(n+1)}$	S	$\dfrac{(n+1)(n+3)}{a^2n^2}\mathrm{e}^{-2nr}[S]^{-(2n+3)/(n+1)}$
$-3 < n < -1$	$a[C]^{1/2(n+1)}$	C	$\dfrac{(n+1)(n+3)}{a^2n^2}\mathrm{e}^{-2nr}[C]^{-(2n+3)/(n+1)}$

In both cases the fluid is irrotational, its equation of state being $p = \mu$, and one has

$$\dot{u} = n\partial_r, \qquad \theta = \mathrm{e}^{-nr}H^{-1}\left(\frac{\dot{H}}{H} + 2\frac{\dot{f}}{f}\right), \qquad \Sigma = \frac{n^2}{(2n+3)^2}. \quad (27)$$

As for the respective dimensionless scalars

$$\Omega = \frac{3(n+1)(n+3)}{n^2(2n+3)^2}\frac{1}{C^2}, \qquad W = \frac{n}{2n+3}\frac{S}{C}, \qquad (28)$$

$$q = \frac{-(4n^2 + 6n + 1)C^2 + 6(n+1)(2n+3)}{(2n+3)^2 C^2}, \qquad (29)$$

for the first case, and

$$\Omega = -\frac{3(n+1)(n+3)}{n^2(2n+3)^2}\frac{1}{S^2}, \qquad W = \frac{n}{2n+3}\frac{C}{S}, \tag{30}$$

$$q = -\frac{(4n^2 + 6n + 1)S^2 + 6(n+1)(2n+3)}{(2n+3)^2 S^2}, \tag{31}$$

for the second case. So, the solutions behave in different ways depending on the value of the parameter n and the range of values of t considered.

Table 7. Case (C). t and r in CI have been re-scaled so as to make $n = 1$.

C	k	X	$A^2(t,r)$	$B^2(t,r)$	$F^2(t,r)$
I	$-1, 0, 1$	$\partial_t + \partial_r$	$e^{t+r}H^2(t-r)$	$e^{t+r}L^2(t-r)$	$e^{t+r}f^2(t-r)$
II	0	$\partial_t + \partial_r - y\partial_y$	$e^{n(t+r)}H^2(t-r)$	$e^{n(t+r)}L^2(t-r)$	$e^{(n+1)(t+r)}f^2(t-r)$

Unfortunately, no general solutions to the field equations for a perfect fluid can be given in these cases, and as it turns out when trying to solve them in particular cases, most of the solutions thus found hold only on some open domains of the manifold of the form $t - r >$ constant, nevertheless we now give two solutions that are valid over the whole spacetime manifold, both of them corresponding to the case CII:

(i) CII, $n = -3$.

$$f = 1, \qquad H^2 = \alpha e^{(\beta-4)(t-r)}, \qquad L^2 = \alpha e^{\beta(t-r)}. \tag{32}$$

Then one has

$$\mu = p = e^{3(t+r)}\alpha^{-1}[(4-\beta)e^{(4-\beta)(t-r)} + \beta e^{-\beta(t-r)}], \tag{33}$$

where α and β are constants which can easily be chosen so as to make $\mu > 0$ all over M. The dimensionless scalars are

$$\Omega = \frac{12}{(\beta-7)^2}\left[4 - \beta + \beta e^{-4(t-r)}\right], \qquad \theta = \frac{\beta-7}{2\sqrt{H}e^{-\frac{3}{2}(t+r)}}, \tag{34}$$

$$W = \frac{1-\beta}{\beta-7}e^{-2(t-r)}, \qquad \Sigma = \frac{(1-\beta)^2}{(\beta-7)^2}. \tag{35}$$

(ii) CII, $n = 1$.

$$f = 1, \qquad H^2 = \alpha L^2, \qquad L^2 = \exp\left[(1-\alpha)(t-r) - \frac{\beta(1-\alpha)}{2}e^{-2(t-r)/(1-\alpha)}\right], \tag{36}$$

and then

$$(\mu - p)e^{t+r}H^2 = 4(1-\alpha), \tag{37}$$

$$(\mu + p)e^{t+r}H^2 = 2(1-\alpha)\left[(1-\alpha) + \beta e^{-2(t-r)/(1-\alpha)}\right], \tag{38}$$

where again α and β are constants. The energy conditions restrict α to values $0 < \alpha < 1$, and if we demand that the solution be valid over the whole manifold, then β must be positive. Notice that in this case, for $\beta \neq 0$, there is no equation of state of the form

$p = p(\mu)$. For $\beta = 0$ the solution is a special case of $\mathcal{H}_5$. The dimensionless scalars are

$$\Omega = \frac{3(1-\alpha)\left[3 - \alpha + \beta e^{-(2/(1-\alpha))(t-r)}\right]}{\left[3 - \frac{1}{2}\alpha + \frac{1}{2}\beta e^{-(2/(1-\alpha))(t-r)}\right]^2},$$

$$\theta = \frac{3 - \frac{1}{2}\alpha + \frac{1}{2}\beta e^{-(2/(1-\alpha))(t-r)}}{\sqrt{H}e^{(t+r)/2}}, \tag{39}$$

$$W = \sqrt{\alpha}\left[-1 + \frac{6}{6 - \alpha + \beta e^{-(2/(1-\alpha))(t-r)}}\right],$$

$$\Sigma = \left[-1 + \frac{6}{6 - \alpha + \beta e^{-(2/(1-\alpha))(t-r)}}\right]^2. \tag{40}$$

In both cases there is no FRW limit.

As a final remark to this section, notice that expressions appearing in tables 2, 4 and 7 are completely general, i.e. valid regardless the material content.

4. The H_3 case

In this section we extend a previous work [34], correct some errors and present new solutions.

The existence of a three-parameter homothetic group H_3, implies that of a $G_2 \subset H_3$ of isometries with orbital dimensions of three and two, respectively (see section 2.1). When the Killing subalgebra has null orbits, the metric is of Kundt's class [50] and perfect fluids are excluded [1]. We shall therefore assume in the following that the Killing orbits are non-null.

A classification of all such spacetimes in terms of the Bianchi type of the homothetic algebra can be found in [34]. See also the references cited therein for a (partial) account of papers on this issue. We can summarize the results concerning the topology of the Killing orbits and the Bianchi type of H_3 in the following table.

Table 8.

G_2-type	G_2-orbits	H_3-Bianchi type
$G_2\mathrm{I}$	$S^1 \times \mathbb{R}$	I, II, III
	$\mathbb{R}^2$	I, II, III, IV, V, VI, VII
$G_2\mathrm{II}$	$\mathbb{R}^2$	III

As the above table shows, two different topologies are possible in the Abelian G_2 case ($G_2\mathrm{I}$) [34] namely, V_2 diffeomorphic to $\mathbb{R}^2$ or to $\mathcal{S}^1 \times \mathbb{R}$; and it follows in the latter case that the only possible Bianchi types for $\mathcal{H}_3$, irrespective of the assumed matter content, are I, II and III (this also holds if the $\mathcal{H}_3$ is replaced by a conformal algebra $\mathcal{C}_3$, see [40]) and for the case $V_2 \cong \mathbb{R}^2$, the seven soluble Bianchi types can occur. For the non-Abelian case the only possible homothetic algebra is of Bianchi type III and its orbits are diffeomorphic to $\mathbb{R}^2$.

4.1. Case G_2 Abelian

In this subsection we shall restrict ourselves to the Abelian case with spacelike isometric orbits diffeomorphic to $\mathbb{R}^2$, giving appropriate 'translation rules' for the other possibilities.

As is customary, we shall assume that the Killing orbits admit orthogonal 2-surfaces (i.e. orthogonally transitive G_2 metrics), since these models can be regarded as spatially inhomogeneous generalizations of the orthogonal Bianchi models of group types I–VII [44], which play an important role in theoretical cosmology. In the previous reference [44] Hewitt and Wainwright have provided a formalism for dealing with this entire class of models from a qualitative point of view. Cosmological models admitting an Abelian G_2 on spacelike orbits have also been studied by Ruiz and Senovilla [51] and Van den Bergh and Skea [53] among others. A major reason for looking at these models with an H_3 is discussed in [44], namely that these models with an H_3 may describe the asymptotic dynamical behaviour at late times or near the big bang for more general G_2 cosmologies. In the non-orthogonally transitive case very little work has been done so far. To the best of our knowledge, the only solutions in Wainwright's classes A(i) and A(ii) so far studied, are those in [54] and [55], respectively.

Adapting two coordinates to two commuting KVs, say $\xi = \partial_x$ and $\eta = \partial_y$, and choosing two other coordinates, t and z, on the surfaces orthogonal to the isometry orbits; it follows that the line element can be written in the form (see, for instance, [56])

$$ds^2 = -A\,dt^2 + B\,dz^2 + R\left[F(dx + W\,dy)^2 + F^{-1}\,dy^2\right], \tag{41}$$

where A, B, R, F and W are all functions of t and z alone.

All the other cases (i.e. timelike Killing orbits and Killing orbits diffeomorphic to $S^1 \times \mathbb{R}$ of either nature, spacelike or timelike) can be obtained formally from the above by means of the substitutions in table 9.

Table 9. φ is the angular coordinate (with the standard periodicity 2π).

$T_2 \cong \mathbb{R}^2$	$V_2 \cong S^1 \times \mathbb{R}$
$\partial_x \mapsto \partial_t$	$y \mapsto \varphi$
$(t, x) \mapsto i(-x, t)$	Regularity condition
$W \mapsto iW$	on the axis

Regarding solutions with an Abelian G_2 acting on timelike orbits (including the astrophysically relevant stationary and axisymmetric models, which have been studied for many years), it is worth mentioning that they have attracted renewed attention, see [38–41].

It is easy to see from the commutation relations of the proper HVF, X, with ξ and η, and the homothetic equation specialized to the components g_{tx}, g_{ty}, g_{zx} and g_{zy} of the metric (41), that X must take the form

$$\begin{aligned} X &= X^t(t, z)\partial_t + X^z(t, z)\partial_z + X^x(x, y)\partial_x + X^y(x, y)\partial_y \\ &\equiv \hat{X} + X^x(x, y)\partial_x + X^y(x, y)\partial_y, \end{aligned} \tag{42}$$

where $X^x(x, y)$ and $X^y(x, y)$ are linear functions of their arguments that yield for every different Bianchi type the forms shown in table 10.

Now, one can always perform a coordinate change in the t, z plane, preserving the forms (41) of the metric and such that $\hat{X}$ is brought to either one of the forms

(i) $\qquad \hat{X} = \partial_t,$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (43)

(ii) $\qquad \hat{X} = \partial_z,$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (44)

(iii) $\qquad \hat{X} = \partial_t + \partial_z.$ $\qquad\qquad\qquad\qquad\qquad\qquad$ (45)

Table 10.

Type	$X^x(x, y)$	$X^y(x, y)$
I	0	0
II	y	0
III	x	0
IV	$x + y$	y
V	x	y
VI	x	qy
VII	$-y$	$x + qy$

The functional form of the metric functions can then be worked out for each Bianchi type, thus for case (i)

$$ds^2 = e^{2nt}\{-A(z)\,dt^2 + B(z)\,dz^2 + e^{\alpha t} R(z)[F(t, z)(dx + W(t, z)\,dy)^2 + F^{-1}(t, z)\,dy^2]\} \tag{46}$$

where n is the homothetic constant. By rescaling the coordinate z one can set $A(z) = B(z)$:

$$(\text{I}) \qquad \alpha = 0, \qquad F = f(z), \qquad W = w(z), \tag{47}$$

$$(\text{II}) \qquad \alpha = 0, \qquad F = f(z), \qquad W = w(z) - t, \tag{48}$$

$$(\text{III}) \qquad \alpha = -1, \qquad F = e^{-t} f(z), \qquad W = e^t w(z), \tag{49}$$

$$(\text{IV}) \qquad \alpha = -2, \qquad F = f(z), \qquad W = w(z) - t, \tag{50}$$

$$(\text{V}) \qquad \alpha = -2, \qquad F = f(z), \qquad W = w(z), \tag{51}$$

$$(\text{VI}) \qquad \alpha = -(1+q), \qquad F = e^{-(1-q)t} f(z), \qquad W = e^{(1-q)t} w(z), \tag{52}$$

$$(\text{VII}) \qquad \alpha = -q, \tag{53}$$

$$F = \frac{2}{\sqrt{4 - q^2}} \left[\sqrt{1 + c(z)^2 + g(z)^2} + c(z) \cos(\sqrt{4 - q^2}\, t) + g(z) \sin(\sqrt{4 - q^2}\, t) \right],$$

$$W = \frac{q}{2} + \frac{(\sqrt{4 - q^2}/2)[-g(z) \cos(\sqrt{4 - q^2}\, t) + c(z) \sin(\sqrt{4 - q^2}\, t)]}{\sqrt{1 + c(z)^2 + g(z)^2} + c(z) \cos(\sqrt{4 - q^2}\, t) + g(z) \sin(\sqrt{4 - q^2}\, t)}.$$

For case (ii) one finds the same expressions as above, by simply reversing the roles of the coordinates t and z, and in case (iii), the correct expressions are those obtained from (46)–(53) after effecting the substitutions $t \longmapsto (t + z)/2$ and $z \longmapsto t - z$.

These coordinates are thus adapted to the homothetic orbits. Notice that the form of these functions holds for any energy–momentum tensor, since no use has been made thus far of the field equations.

For perfect fluid solutions admitting an orthogonally transitive G_2 on spacelike orbits (metric (41)), it is customary to use coordinates adapted to the fluid flow and it is easy to see (see theorems 3.1 and 3.2 in [56]) that one can always perform a coordinate change in the t, z plane so as to bring the fluid flow to a comoving form, preserving the form of the metric (41) (i.e. $u = \frac{1}{\sqrt{A}} \frac{\partial}{\partial t}$). Obviously, the form of the proper HVF X will change and the coordinates will no longer be adapted to the homothetic orbits; however, equation (6) along with the remaining coordinate freedom in the t, z plane ($t \to m(t), z \to n(z)$) allow us to bring the HVF to the form (42) with $\hat{X}$ as in (43)–(45); the metric functions thus being the ones previously found.

Thus, three classes of perfect fluid solutions arise, depending on the orientation of the fluid flow u relative to the homothetic orbits:

(i) The fluid flow is tangent to the homothetic orbits and they are then timelike.

(ii) The fluid flow is orthogonal to the homothetic orbits and therefore they are spacelike.

(iii) 'Tilted' fluid flow, i.e. u is neither tangential nor orthogonal to the homothetic orbits, which are then not constrained *a priori* to being timelike or spacelike, and hence their nature may vary from point to point over the spacetime.

4.1.1. Fluid flow tangent to the homothetic orbits. This case, assuming the existence of two hypersurface orthogonal KVs (i.e. diagonal metric) has been studied thoroughly by Wainwright, Hewitt and collaborators in an interesting series of articles [42–44], where the properties of these models are analysed using the qualitative theory of plane autonomous systems, showing that (first-class) self-similar solutions within this family can represent the asymptotic states at later times of more general inhomogeneous G_2 models. Uggla [35] found four explicit solutions of this type. Here we shall not give any explicit solution belonging to this class (see the above references and those cited therein), but rather provide the general form of the metric functions (including the non-diagonal cases) and briefly discuss the generic behaviour of the kinematical quantities associated with the fluid (acceleration, deceleration parameter, shear, etc).

The possible forms for the metric functions appearing in (41) are precisely those given in (46)–(53).

The cases studied by Wainwright and collaborators correspond to types I, III, V and VI, since these are the only ones in which the function W can be set equal to zero. For type VII, $W = q/2$ implies the existence of a further Killing vector tangent to the Killing orbits and the metric would then admit a multiply transitive group H_4 of homotheties. Notice also, that the diagonal cases are separable in the variables t, z, thus being special cases of the solutions studied by Ruiz and Senovilla [51] or those of Agnew and Goode [52] for $\gamma = 2$.

Specializing equation (6) to the matter variables μ and p, we obtain

$$\mu = \mathrm{e}^{-2nt} \hat{\mu}(z), \qquad p = \mathrm{e}^{-2nt} \hat{p}(z), \tag{54}$$

and computing the kinematical quantities associated with the fluid velocity vector for the metric (46)

$$\theta = \frac{3n + \alpha}{\mathrm{e}^{nt}\sqrt{A}}, \qquad q = \frac{-\alpha}{3n + \alpha}, \qquad \dot{u} = \frac{1}{2}\frac{A'}{A}\mathrm{d}z, \tag{55}$$

where a prime denotes a derivative with respect to z and q here denotes the deceleration parameter. The non-vanishing components of the shear tensor are

$$\sigma_{zz} = -\mathrm{e}^{nt}\sqrt{A}\frac{\alpha}{3}, \qquad \sigma_{xx} = \mathrm{e}^{nt}\frac{RF}{6\sqrt{A}}\left(3\frac{\dot{F}}{F} + \alpha\right), \tag{56}$$

$$\sigma_{xy} = \mathrm{e}^{nt}\frac{RF}{6\sqrt{A}}\left(3W\frac{\dot{F}}{F} + W\alpha + 3\dot{W}\right), \tag{57}$$

$$\sigma_{yy} = \frac{\mathrm{e}^{nt}}{6\sqrt{A}}\frac{R}{F}\left(-3\frac{\dot{F}}{F} + \alpha + 3W^2 F\dot{F} + W^2 F^2 \alpha + 6F^2 W\dot{W}\right), \tag{58}$$

thus, the shear scalar is

$$\sigma^2 = \frac{3(\dot{F}/F)^2 + \alpha^2 + 3F^2\dot{W}^2}{12A\mathrm{e}^{2nt}}. \tag{59}$$

If one assumes that the fluid has an equation of state of the form (7), from the contracted Bianchi identities it follows that

$$\gamma(\dot{\mu} + \mu\theta)u^a + \gamma\mu\dot{u}^a + (\gamma - 1)\mu_{,b}g^{ba} = 0, \tag{60}$$

where $\dot{\mu} \equiv \mu_{,t} u^t$. Contracting the above expression with u^a, one gets

$$\dot{\mu} = -\gamma \mu \theta. \tag{61}$$

Assuming $\theta \neq 0$ (the case $\theta = 0$, although mathematically possible, is not physically interesting since it would correspond to a non-expanding universe, thus contradicting observations) and substituting (54) and (55) in equation (61), it follows that

$$\gamma = \frac{2n}{3n + \alpha}. \tag{62}$$

Specializing the quantities γ and q to each Bianchi type, one gets the values given in table 11.

Table 11.

Type	I	II	III	IV	V	VI	VII
γ	$\frac{2}{3}$	$\frac{2}{3}$	$\dfrac{2n}{3n-1}$	$\dfrac{2n}{3n-2}$	$\dfrac{2n}{3n-2}$	$\dfrac{2n}{3n-(1+q)}$	$\dfrac{2n}{3n-q}$
q	0	0	$\dfrac{1}{3n-1}$	$\dfrac{2}{3n-2}$	$\dfrac{2}{3n-2}$	$\dfrac{1+q}{3n-(1+q)}$	$\dfrac{q}{3n-q}$

Notice that the only shear-free solution is the type I one. For types II–VI there is no limit where $\Sigma \equiv 3\sigma^2/\theta^2$ becomes zero and for type VII, $\Sigma \to 0$ for some spatial limit if and only if $c(z) \to 0$, $g(z) \to 0$ and $q = 0$ (i.e. $F \to 1$ and $W \to 0$). Thus, only types I and VII can have solutions with a FRW limit, but in both cases $\mu + 3p = 0$ and $q = 0$, so these models are not physically relevant at the present time but they may be relevant before the decoupling time and/or in the future.

4.1.2. Fluid flow orthogonal to the homothetic orbits. The case of spatially homothetic orbits was thoroughly studied by Eardley [10] where a classification scheme of these models was given and their dynamical properties were studied. Luminet [57] constructed a convenient basis of 1-forms and gave its explicit form in terms of a standard coordinate basis $\{dx^a\}$ as well as the expression of the homothetic vector in the dual basis $\{\partial/\partial x^a\}$. He also proved a theorem showing that perfect fluid models of a certain class were incomplete in the sense of Hawking and Ellis [58].

For the sake of completeness, we just mention that the form of the metric, in this case, can be formally obtained from (46)–(53) by simply reversing the roles of the coordinates t and z.

The expressions of the acceleration, expansion, deceleration parameter and shear scalar are given by

$$\dot{u} = n\, dz, \qquad \theta = \frac{1}{2e^{nz}\sqrt{A}} \left\{ \frac{\dot{A}}{A} + 2\frac{\dot{R}}{R} \right\}, \tag{63}$$

$$q = \frac{2[4(\dot{A}/A)^2 + \dot{A}\dot{R}/AR + 4(\dot{R}/R)^2 - 3\ddot{A}/A - 6\ddot{R}/R]}{(\dot{A}/A + 2\dot{R}/R)^2}, \tag{64}$$

$$\sigma^2 = \frac{(\dot{A}/A - \dot{R}/R)^2 + 3(\dot{F}/F)^2 + 3F^2(\dot{W})^2}{12Ae^{2nz}}. \tag{65}$$

In this case, the homothetic vector field, X, and the 4-velocity u are mutually orthogonal, thus if a barotropic equation of state is assumed, then necessarily $p = \mu$, i.e. stiff-matter

and by simple inspection of the field equations this is seen to be equivalent to

$$\frac{\ddot{R}}{R} = (2n + \alpha)^2. \tag{66}$$

4.1.3. 'Tilted' fluid flow. Finally, the form (iii) for the proper homothetic vector field is precisely the case we are currently interested in, namely, u not tangent or orthogonal to the homothetic orbits.

Specializing now the homothetic equation to the metric (41) we obtain

$$ds^2 = e^{n(t+z)} \left\{ -A(t-z)\, dt^2 + B(t-z)\, dz^2 \right. $$
$$\left. + e^{\alpha(t+z)/2} R(t-z) \left[F(dx + W\, dy)^2 + F^{-1}\, dy^2 \right] \right\}, \tag{67}$$

where again n is the homothetic constant. The parameter α and the functional form of the metric functions for each Bianchi type are those given by (47)–(53) after effecting the following substitutions (see section 4.1)

$$t \longmapsto \frac{t+z}{2}, \qquad z \longmapsto t-z. \tag{68}$$

The kinematical quantities are

$$\dot{u} = \frac{1}{2}\left(n - \frac{A'}{A}\right) dz, \qquad \theta = \frac{3n + \alpha + B'/B + 2R'/R}{2e^{n(t+z)/2}\sqrt{A}}, \tag{69}$$

$$\sigma^2 = \frac{(\alpha/2 - B'/B + R'/R)^2 + 3(F_{,t}/F)^2 + 3F^2(W_{,t})^2}{12e^{n(t+z)}A}, \tag{70}$$

where a prime denotes a derivative with respect to $(t-z)$. A careful study shows that there are no shear-free solutions in this case admitting a maximal group H_3 of homotheties: shear-free solutions are not possible for types II and IV; for the Bianchi types I, V and VII, the shear-free condition implies that the functions F and W must be constants and therefore a further Killing vector tangent to the Killing orbits occurs; for types III and VI, W must vanish, the type III solution then being a homogeneous Bianchi VI model and the type VI one such that $\mu + p = 0$, thus not corresponding to a perfect fluid.

Notice that in [34], the homothetic constant n was set equal to 1 from the beginning, before choosing the coordinates in the surfaces orthogonal to the Killing orbits. By doing so, some solutions were left out, since, although one can always rescale the proper HVF X with a factor $1/n$, such a scaling cannot always be reabsorbed by a redefinition of the coordinates. We correct that error here.

As was pointed out previously, all diagonal ($W = 0$), perfect fluid solutions (admitting an orthogonally transitive Abelian G_2 with flat spacelike orbits) and such that the metric functions A, B, R and F are separable in the variables t and z are already known [51, 52].

Next we shall present some new exact solutions not included in [34], which have been obtained assuming $W = 0$ (diagonal), but which are not of separable variables in the above sense.

Type III:

$$R = 1, \qquad f = e^{\lambda(t-z)}, \tag{71}$$
$$\frac{A}{B} = \frac{1 - 2\lambda}{1 + 2\lambda}, \qquad A = a \exp\left(n(t-z) + \frac{\alpha}{\frac{1}{2} - n} e^{(\frac{1}{2} - n)(t-z)}\right),$$

where $0 < \lambda < \frac{1}{2}$ and $\lambda^2 + n^2 = \frac{1}{2}$ and

$$(\mu - p)Ae^{n(t+z)} = \frac{4\lambda}{1 + 2\lambda}\left(\frac{1}{2} - n\right)^2,$$

$$(\mu + p)Ae^{n(t+z)} = \frac{4\lambda}{1 + 2\lambda}\left(n - \frac{1}{2}\right)\frac{A'}{A},$$

and λ, α and a are constants. In order to satisfy the energy conditions, if $n > 0$, α must also be positive and the value of n is then restricted to $(1/2, \sqrt{2}/2)$; for negative values of n, α is also negative and $n \in (-\sqrt{2}/2, -1/2)$, in these cases a barotropic equation of state does not exist. When $\alpha = 0$ the solution is a particular case of a homogeneous Bianchi VI model.

Type III:

$$R = 1, \qquad f = e^{\lambda(t-z)},$$

$$\frac{A}{B} = e^{(2n-1)(t-z)}, \qquad \lambda^2 = (n-1)^2, \qquad n \neq \frac{1}{2},$$

$$B = a\exp\left(-\frac{\lambda + 2(n-1)^2}{2n-1}(t-z)\right)\left[\exp(-(2n-1)(t-z)) - 1\right]^c,$$
(72)

where a and c are constants

$$\mu = p = \frac{1}{2A}\left[2n - \frac{3+c}{2}\right]e^{-n(t+z)}.$$

Type III:

$$R = e^{\lambda(t-z)}, \qquad f = e^{-\frac{1}{2}(t-z)}, \qquad A = a^2e^{(2\lambda+2n-1)(t-r)}, \qquad B = b^2,$$
(73)

where a and b are constants, $\lambda = -(2n-1) + \sqrt{3n^2 - 2n + 1/4}$, and

$$(\mu - p)e^{n(t+z)} = \frac{1}{B}\lambda(2n - 1 - 2\lambda),$$

$$(\mu + p)e^{n(t+z)} = \frac{1}{A}(n^2 + \frac{1}{2}\lambda - \frac{3}{4}) + \frac{1}{B}\lambda(\lambda + 2n - \frac{3}{2}).$$

From where it follows that, in order that the solution satisfies the weak and dominant energy conditions all over the manifold, one must have

$$n \in (0.821\,036\,816\,240\,750\,1\ldots, \tfrac{3}{2}),$$

and, again, a barotropic equation of state $p = p(\mu)$ does not exist, except in the case $n = \frac{3}{2}$, when $\mu = p$.

Type V:

$$ds^2 = \frac{e^{t+z}}{f_0^2}|\varphi|^{2c^2+2c}\left\{\frac{-M^4\varphi^2\,dt^2 + dz^2}{M^4\varphi^2 - 1}\right\} + |\varphi|^{-2c}\,dx^2 + |\varphi|^{2c+2}\,dy^2,$$

$$\mu = p = \frac{f_0^2}{e^{t+z}}\frac{M^4\varphi^2 - 1}{2M^2|\varphi|^{2c^2+2c+2}}, \qquad n = 1,$$
(74)

where c, M and f_0 are constants and φ is a function of $t - z$ given implicitly by

$$M^2(t - z) = \ln|\varphi| - \frac{M^4}{2}\varphi^2.$$

Type V:

$$R = 1, \qquad f = e^{\lambda(t-z)}, \tag{75}$$

$$\frac{A}{B} = e^{2(n-1)(t-z)}, \qquad \lambda^2 = (n-1)(n-3),$$

$$A = a e^{(n+1)(t-z)} \left[1 - e^{-2(n-1)(t-z)}\right]^{c/2(n-1)},$$

$$p = \mu = \frac{(n-1)(4-c)}{2A} e^{-n(t+z)},$$

with a and c constants, and n restricted to being $n < 1$ or $n > 3$ in order to satisfy the energy conditions.

For type VI two solutions have been obtained. For the sake of simplicity, we will give them in non-comoving coordinates. Thus

$$ds^2 = \frac{e^{2t}}{F^2} \left\{ -dt^2 + dz^2 + e^{-2t} B^2 \, dx^2 + e^{-2qt} S^2 \, dy^2 \right\}, \tag{76}$$

$$u_t = -\frac{e^t}{F} \cosh a, \qquad u_z = \frac{e^t}{F} \sinh a. \tag{77}$$

The first solution is

$$F = f_0 B, \qquad S = C^{1-q \sin c/(1-q)} E, \qquad B = C^{-q \sin c/(1-q)} E, \tag{78}$$

$$\mu = p = e^{-2t} 2q(1-q) f_0^2 \alpha C^{-2-2q \sin c/(1-q)} (A \cos c + 2\alpha \sin c) E^2,$$

where

$$A = \alpha^2 e^{(1-q)z} - e^{-(1-q)z}, \qquad C = \alpha^2 e^{(1-q)z} + e^{-(1-q)z},$$

$$E = \exp \left[\frac{2q \cos c}{1-q} \tan^{-1} \left(\alpha e^{(1-q)z} \right) \right],$$

and

$$\cosh a = \frac{\sqrt{1 + \sin c} \, C}{\sqrt{4\alpha A \cos c + 8\alpha^2 \sin c}}, \qquad \sinh a = \frac{-(1 + \sin c) A + 2\alpha \cos c}{\sqrt{1 + \sin c} \sqrt{4\alpha A \cos c + 8\alpha^2 \sin c}},$$

where c, α and f_0 are constants. In order to have positive energy density, the parameter q is restricted to $q \in (0, 1)$. A particular, simpler case can be obtained by choosing $\cos c = 0$ and $\sin c = 1$.

The other solution is

$$F = f_0 B, \qquad S = R^{1+q^2/c(1-q)} T^{1+c/(1-q)}, \qquad B = R^{q^2/c(1-q)} T^{c/(1-q)},$$

$$\mu = p = e^{-2t} \frac{q-1}{c} \alpha^2 f_0^2 \left[c^2 R^2 - q^2 T^2 \right] R^{2q^2/c(1-q)-2} T^{2c/(1-q)-2}, \tag{79}$$

where

$$R = e^{\frac{1}{2}(1-q)z} - \alpha^2 e^{-\frac{1}{2}(1-q)z}, \qquad T = e^{\frac{1}{2}(1-q)z} + \alpha^2 e^{-\frac{1}{2}(1-q)z},$$

and

$$\cosh a = \frac{(c-q) T R}{2\alpha \sqrt{c^2 R^2 - q^2 T^2}}, \qquad \sinh a = \frac{q T^2 - c R^2}{2\alpha \sqrt{c^2 R^2 - q^2 T^2}},$$

where again c, α and f_0 are constants. Notice that the solution is only valid for

$$c^2 R^2 - q^2 T^2 > 0.$$

Notice that the solutions that have a stiff-matter equation of state ($p = \mu$) can be derived from vacuum solutions (also admitting an Abelian G_2) using a method proposed by Wainwright *et al* [59].

4.2. Non-Abelian G_2

For a non-Abelian G_2, a local system of coordinates can be chosen in which the Killing vectors, say ξ and η, are

$$\xi = \partial_1, \qquad \eta = x^1 \partial_1 + \partial_2. \tag{80}$$

Now, supposing the existence of a proper homothetic vector field, say X, one can see that the only allowed Bianchi type for H_3 is III, i.e.

$$[\xi, \eta] = \xi, \qquad [\xi, X] = 0, \qquad [\eta, X] = 0. \tag{81}$$

Taking into account (80) and (81) one easily comes to the following form of X:

$$X^a = (e^{x^2} X^1(x^3, x^4), X^2(x^3, x^4), X^3(x^3, x^4), X^4(x^3, x^4)). \tag{82}$$

Note that, since the orbits associated with H_3 and G_2 cannot coincide [8], the components X^3 and X^4 of the homothetic vector field cannot both vanish. Also, notice that the homothetic constant can in this case be set equal to one without altering our choice of coordinates.

4.2.1. The orthogonally transitive case. As in the previous Abelian case, we will restrict our attention just to the orthogonally transitive G_2 metrics. Regarding non-orthogonally transitive G_2, a discussion can be found in [60] where a study of perfect fluid solutions with 4-velocity orthogonal to the isometric orbits is given. In that reference it is also assumed that the Killing vector ξ is hypersurface orthogonal and the homothetic vector field, X, is orthogonal to the fluid velocity. These assumptions imply that the fluid is to be 'stiff' ($p = \mu$) without any *a priori* assumption of an equation of state. No explicit solutions are known so far, but it is shown that solutions with pressure and matter positive on an open set can in principle exist by suitably specifying the initial conditions.

For orthogonally transitive G_2, the metric can be written as

$$g_{ab} = \begin{pmatrix} e^{-2x^2} a_{11} & e^{-x^2} a_{12} & 0 & 0 \\ e^{-x^2} a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{83}$$

where $\epsilon = \pm 1$ and $a_{ij} = a_{ij}(x^3, x^4)$.

For this case, on can see that, assuming non-null homothetic orbits V_3, it is always possible to perform a coordinate change in the 2-spaces orthogonal to the Killing orbits, such that it brings the homothetic vector field to the form

$$X^a = (e^{x^2} X^1(x^3, x^4), X^2(x^3, x^4), 1, 0), \tag{84}$$

and the line element can be written as

$$ds^2 = A(e^{-x^2} dx^1 + W dx^2)^2 + B(dx^2)^2 + F\left((dx^3)^2 + \epsilon(dx^4)^2\right), \tag{85}$$

where A, B, F and W are functions of x^3 and x^4 alone.

Now specializing the homothetic equation to the homothetic vector (84) and the metric (85) yields the following forms for X and the metric functions:

$$X^a = (\alpha e^{x^2}, n, 1, 0), \qquad \alpha, n = \text{constant}, \tag{86}$$

$$A = e^{2(1+n)x^3} a(x^4), \qquad B = e^{2x^3} b(x^4), \qquad F = e^{2x^3} f(x^4), \tag{87}$$

and

$$W = \begin{cases} -\dfrac{\alpha}{n} + e^{-nx^3} w(x^4) & n \neq 0 \\ -\alpha x^3 + w(x^4) & n = 0. \end{cases} \tag{88}$$

In the case $n \neq 0$, one can still perform the coordinate change

$$\hat{x}^1 = x^1 - \frac{\alpha}{n} e^{x^2}, \tag{89}$$

so that the homothetic vector field and W take the forms

$$X^a = (0, n, 1, 0), \qquad W = e^{-nx^3} w(x^4). \tag{90}$$

It can be easily shown, just by computing the Einstein and Riemann tensors, that the only vacuum solution with a non-Abelian group of isometries acting orthogonally transitively and admitting a proper homothetic vector field is Minkowski spacetime.

For perfect fluid solutions, equation (6) specialized to the Killing vectors (80) implies for the fluid velocity

$$u_a = \left(e^{-2} u_1(x^3, x^4), u_2(x^3, x^4), u_3(x^3, x^4), u_4(x^3, x^4)\right). \tag{91}$$

For orthogonally transitive G_2 models, it is possible to perform a change of coordinates in the x^3, x^4 plane so as to write the 4-velocity and the metric as

$$u_a = \left(e^{-x^2} \tilde{u}_1(x^3, x^4), \tilde{u}_2(x^3, x^4), 0, \tilde{u}_4(x^3, x^4)\right), \tag{92}$$

$$ds^2 = A(x^3, x^4)(e^{-x^2} dx^1 + W(x^3, x^4) dx^2)^2 + B(x^3, x^4)(dx^2)^2$$
$$+ F(x^3, x^4)(dx^3)^2 + G(x^3, x^4)(dx^4)^2, \tag{93}$$

and, as a consequence, the field equations take on a much simpler form. In this coordinate chart and taking into account (6) and (1) specified to the components g_{13}, g_{14}, g_{23} and g_{24}, it is easy to see that X must be of the form

$$X^a = (\alpha e^{x^2}, n, X^3(x^3), X^4(x^4)), \tag{94}$$

and the following possibilities then arise:

I $\qquad X^a = (\alpha e^{x^2}, n, 0, 1),$ $\qquad\qquad\qquad\qquad\qquad$ (95)

II $\qquad X^a = (\alpha e^{x^2}, n, 1, 0),$ $\qquad\qquad\qquad\qquad\qquad$ (96)

III $\qquad X^a = (\alpha e^{x^2}, n, 1, 1).$ $\qquad\qquad\qquad\qquad\qquad$ (97)

In the three cases, for $n \neq 0$ one can perform the coordinate change (89) thus enabling one to set α zero.

The homothetic equation specialized to the metric (93) then yields the following possibilities.

Table 12. Subcases a and b refer to whether $n \neq 0$ or $n = 0$ respectively, and v stands for $x^3 - x^4$.

Type	X^a	$A(x^3, x^4)$	$W(x^3, x^4)$
I.a	$(0, n, 0, 1)$	$e^{2(1+n)x^4} a(x^3)$	$e^{-nx^4} w(x^3)$
I.b	$(\alpha e^{x^2}, 0, 0, 1)$	$e^{2x^4} a(x^3)$	$-\alpha x^4 + w(x^3)$
II.a	$(0, n, 1, 0)$	$e^{2(1+n)x^3} a(x^4)$	$e^{-nx^3} w(x^4)$
II.b	$(\alpha e^{x^2}, 0, 1, 0)$	$e^{2x^3} a(x^4)$	$-\alpha x^3 + w(x^4)$
III.a	$(0, n, 1, 1)$	$e^{(1+n)(x^3+x^4)} a(v)$	$e^{-\frac{1}{2} n(x^3+x^4)} w(v)$
III.b	$(\alpha e^{x^2}, 0, 1, 1)$	$e^{x^3+x^4} a(v)$	$-\frac{1}{2}\alpha(x^3 + x^4) + w(v)$

Table 13.

Type	$B(x^3, x^4)$	$F(x^3, x^4)$	$G(x^3, x^4)$
I	$e^{2x^4} b(x^3)$	$e^{2x^4} f(x^3)$	$\epsilon e^{2x^4} f(x^3)$
II	$e^{2x^3} b(x^4)$	$e^{2x^3} f(x^4)$	$\epsilon e^{2x^3} f(x^4)$
III	$e^{x^3+x^4} b(v)$	$e^{x^3+x^4} f(v)$	$e^{x^3+x^4} g(v)$

Table 14.

Type	$\tilde{u}_1(x^3, x^4)$	$\tilde{u}_2(x^3, x^4)$	$\tilde{u}_4(x^3, x^4)$
I.a	$e^{(1+n)x^4} \hat{u}_1(x^3)$	$e^{x^4} \hat{u}_2(x^3)$	$e^{x^4} \hat{u}_4(x^3)$
I.b	$e^{x^4} \hat{u}_1(x^3)$	$-\alpha x^4 e^{x^4} \hat{u}_1(x^3) + e^{x^4} \hat{u}_2(x^3)$	$e^{x^4} \hat{u}_4(x^3)$
II.a	$e^{(1+n)x^3} \hat{u}_1(x^4)$	$e^{x^3} \hat{u}_2(x^4)$	$e^{x^3} \hat{u}_4(x^4)$
II.b	$e^{x^3} \hat{u}_1(x^4)$	$-\alpha x^3 e^{x^3} \hat{u}_1(x^4) + e^{x^3} \hat{u}_2(x^4)$	$e^{x^3} \hat{u}_4(x^4)$
III.a	$e^{\frac{1}{2}(1+n)(x^3+x^4)} \hat{u}_1(v)$	$e^{\frac{1}{2}(x^3+x^4)} \hat{u}_2(v)$	$e^{\frac{1}{2}(x^3+x^4)} \hat{u}_4(v)$
III.b	$e^{\frac{1}{2}(x^3+x^4)} \hat{u}_1(v)$	$-\frac{1}{2}\alpha(x^3 + x^4) e^{\frac{1}{2}(x^3+x^4)} \hat{u}_1(v) + e^{\frac{1}{2}(x^3+x^4)} \hat{u}_2(v)$	$e^{\frac{1}{2}(x^3+x^4)} \hat{u}_4(v)$

Since $u_3 = 0$ in this coordinate chart, the components of the Einstein tensor G_{13}, G_{23} and G_{34} must vanish identically, hence

$$W_{,3} = 0, \tag{98}$$

$$\frac{A_{,3}}{A} = \frac{B_{,3}}{B}, \tag{99}$$

$$0 = \frac{B_{,3}}{B}\left(-\frac{1}{4}\frac{A_{,4}}{A} + \frac{1}{2}\frac{F_{,4}}{F} + \frac{3}{4}\frac{B_{,4}}{B}\right) + \frac{1}{4}\frac{G_{,3}}{G}\left(\frac{A_{,4}}{A} + \frac{B_{,4}}{B}\right) - \frac{B_{,34}}{B}, \tag{100}$$

where $_{,i}$ means a derivative with respect x^i. From these equations we found more explicit forms for the metric functions, that are given by

Case I(a).

$$ds^2 = e^{2x^4}\left\{k e^{2nx^4}\left[f(x^3)\right]^{(2+n)/n}\left(e^{-x^2} dx^1 + w e^{-nx^4} dx^2\right)^2 + b\left[f(x^3)\right]^{(2+n)/n}\left(dx^2\right)^2 \right.$$
$$\left. + f(x^3)\left(\left(dx^3\right)^2 + \epsilon\left(dx^4\right)^2\right)\right\}, \tag{101}$$

where $k = \pm 1$, w and b are arbitrary constants. Note that n must be different from zero or a third Killing vector occurs and the metric becomes then LRS. In such a case the metric functions $a(x^3)$ and $b(x^3)$ are still proportional to each other, but no relation exits, in principle, with $f(x^3)$.

Case I(b).

$$ds^2 = e^{2x^4}\left\{a(x^3)\left(e^{-x^2} dx^1 - \alpha x^4 dx^2\right)^2 + b a(x^3)\left(dx^2\right)^2 + \left(dx^3\right)^2 + \epsilon\left(dx^4\right)^2\right\}, \tag{102}$$

where b is an arbitrary constant different from zero to prevent a non-singular metric, and $\alpha \neq 0$ if the group G_2 is to be maximal.

Case II.

Only one possibility arises in this case, namely $\alpha = n = 0$; thus the line element becomes

$$ds^2 = e^{2x^3}\left\{a(x^4)\left(e^{-x^2}\,dx^1 + w(x^4)\,dx^2\right)^2 + b(x^4)\left(dx^2\right)^2 + \left(dx^3\right)^2 + \epsilon\left(dx^4\right)^2\right\}, \tag{103}$$

Case III(a).

$$ds^2 = e^{x^3+x^4}\left\{e^{n(x^3+x^4)}a(v)\left(e^{-x^2}\,dx^1 + we^{-nx^4}\,dx^2\right)^2\right.$$
$$\left. + be^{n(x^3-x^4)}a(v)\left(dx^2\right)^2 + f(v)\left(dx^3\right)^2 + g(v)\left(dx^4\right)^2\right\}, \tag{104}$$

where w is an arbitrary constant and $v = x^3 - x^4$.

Case III(b).

$$ds^2 = e^{x^3+x^4}\left\{a(v)\left[\left(e^{-x^2}\,dx^1 - \alpha x^4\,dx^2\right)^2 + b\left(dx^2\right)^2\right] + f(v)\left(dx^3\right)^2 + g(v)\left(dx^4\right)^2\right\}. \tag{105}$$

Again, α and n are arbitrary non-null constants. The differential equation (100) for case III can be rewritten as

$$0 = \frac{g'}{g}\left(1 - \frac{a'}{a}\right) - \frac{f'}{f}\left(1 + n + \frac{a'}{a}\right) + 1 + n\frac{a'}{a} + \left(\frac{a'}{a}\right)^2 + 2\left(\frac{a'}{a}\right)', \tag{106}$$

where the prime denotes an ordinary derivative with respect to the variable $v = x^3 - x^4$.

4.2.2. Diagonal case. Since the field equations for a perfect fluid are still so complicated we will make a further assumption that will bring the metric into diagonal form; namely, the Killing vector ξ being hypersurface orthogonal.

The possibilities are now restricted just to diagonal subcases (a), since diagonal subcases (b) do always admit a further Killing vector tangent to the Killing orbits V_2.

Computing the Einstein tensor for those metrics, one has

$$G_{14} = 0 \qquad \text{and} \qquad G_{24} \neq 0. \tag{107}$$

Consequently, we have chosen a coordinate chart in such a way that the fluid flow velocity always lies in the 2-plane spanned by $\partial/\partial x^2$ and $\partial/\partial x^4$ at each point. Therefore, in all cases we will have

$$u = u_2\,dx^2 + u_4\,dx^4, \qquad \frac{(u_2)^2}{g_{22}} + \frac{(u_4)^2}{g_{44}} = -1. \tag{108}$$

A careful analysis of all the possibilities reveals that, in most cases, there exist further KVs and, in the instance of null homothetic orbits, the energy conditions cannot be fulfilled. Apart from these cases, it is worth mentioning that whenever X is orthogonal to u, the metric and field equations are

$$ds^2 = e^{2x^3}\left\{a^2(x^4)e^{-2x^2}\left(dx^1\right)^2 - \epsilon b^2(x^4)\left(dx^2\right)^2 + \left(dx^3\right)^2 + \epsilon\left(dx^4\right)^2\right\}, \tag{109}$$

$$0 = 2\epsilon - \frac{1}{b^2} + \frac{a'b'}{ab} + \frac{a''}{a}, \tag{110}$$

$$0 = \left[\frac{a'}{a} - \frac{b'}{b}\right]^2 - \left[\frac{b''}{b} - \frac{a''}{a}\right]\left[-\frac{1}{b^2} + \frac{a'b'}{ab} - \frac{b''}{b}\right], \tag{111}$$

with a prime indicating a derivative with respect to (x^4), and then one necessarily has

$$p = \mu = e^{-2x^3} \left\{ 1 + \epsilon \frac{b''}{b} \right\}, \tag{112}$$

without previously assuming the existence of a barotropic equation of state.

Acknowledgments

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