

The gravitational and electrostatic fields far from an isolated Einstein-Maxwell source

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Abstract

The exterior solution for an arbitrary charged, massive source, is studied as a static deviation from the Reissner-Nordström metric. This is reduced to two coupled ordinary differential equations for the gravitational and electrostatic potential functions. The homogeneous equations are explicitly solved in the particular case $q^2 = m^2$, obtaining a multipole expansion with radial hypergeometric dependence for both potentials. In the limiting case of a neutral source, the equations are shown to coincide with recent results by Bondi and Rindler.

I. INTRODUCTION

A series of well-known and elegant results in general relativity (see references in [1]), show that the multipole moments of an isolated, stationary source determine in a complete and unique manner the gravitational field on its surrounding 3-space. This has been also extended to the case of charged sources, both for static [2] and stationary [3] cases. Recently however, Bondi and Rindler stated the convenience of obtaining precise expressions for the

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multipole expansion of a fully- relativistic isolated source by means of a static perturbation defined over the Schwarzschild metric. In this way, comparison between newtonian gravity and general relativity can be achieved in quantitative terms. In the same spirit, this work presents an extension *à la* Bondi and Rindler to the Einstein-Maxwell case. Thus, we consider a bounded and static, but otherwise arbitrary distribution of charged matter. The assumed static character ensures that the only conserved quantities for such a system are the total gravitational mass-energy m and the net charge q , as evaluated by an observer situated well away from the source. It is sensible to expect that such an observer would find a space-time not markedly different from the Reissner-Nordstrøm one. As Bondi and Rindler would say, “the Reissner-Nordstrøm metric gives bones to the space”. We therefore propose a description of the exterior geometry as a small static deviation from the Reissner-Nordstrøm solution.

This paper is organized as follows: in section **II**, the relevant Einstein-Maxwell equations are presented, and the constraints imposed by the field equations on the perturbation terms are found. In section **III**, the physical implications of the results are interpreted. The particular cases $q^2 = m^2$ and $q = 0$ are discussed in detail.

II. THE EINSTEIN-MAXWELL FIELD EQUATIONS

In curvature coordinates (t, r, θ, φ) , and taking advantage of the static character of the system so as to eliminate the mixed time-space components of the metric tensor, and to diagonalize its pure spatial part by means of a coordinate transformation, we are allowed to adopt the following general *ansatz* for the line element:

$$ds^2 = (1 + 2\alpha) p dt^2 - (1 + 2\beta) \frac{1}{p} dr^2 - (1 + 2\gamma) r^2 d\theta^2 - (1 + 2\delta) r^2 \sin^2 \theta d\varphi^2, \quad (1)$$

where

$$p(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \quad (2)$$

is the usual Reissner-Nordström metric function, and α , β , γ and δ are small functions of the spatial coordinates. On the other hand, the vector potential for the electric field will be chosen as

$$A_a = \left(\frac{q}{r} - \Phi \right) \delta_a^0, \quad (3)$$

with Φ defined again as a small function of the spatial coordinates. It is obvious from (1) and (3) that the five-parameter set of perturbations α , β , γ , δ and Φ represent all possible non-spherical terms for the physical fields, and that their vanishing brings us back to the spherically symmetric Reissner-Nordström solution. This set of functions, defined in a somewhat similar manner to [1], is also completely equivalent to the time-independent part of Chandrasekhar's polar perturbations [4], when considered as first-order quantities. Their behaviour is governed by the electrovacuum Einstein-Maxwell field equations,

$$\begin{aligned} R_{ab} &= 8\pi T_{ab} \\ &= 2F_{ac}F^c_b + \frac{1}{2}g_{ab}F_{cd}F^{cd}, \\ \nabla_b F^{ab} &= 0, \end{aligned} \quad (4)$$

where $F_{ab} = 2\nabla_{[a}A_{b]}$ is the electromagnetic Maxwell-Faraday tensor. Since we are interested in the far-field solutions, we can neglect squares and products of the perturbations, as well as of their derivatives, so that the non-vanishing Einstein-Maxwell equations become a set of eight partial differential equations, which we transcribe below:

$$(R_{tt} = 8\pi T_{tt}):$$

$$\begin{aligned} &\frac{\partial^2 \alpha}{\partial r^2} + \frac{2}{r} \frac{\partial \alpha}{\partial r} + \frac{p'}{2p} \frac{\partial}{\partial r} (3\alpha - \beta + \gamma + \delta) + \frac{2q^2}{pr^4} (1 + 2\alpha) \\ &+ \frac{1}{pr^2} \left(\frac{\partial^2 \alpha}{\partial \theta^2} + \cot \theta \frac{\partial \alpha}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \alpha}{\partial \varphi^2} \right) \\ &= -\frac{2q}{pr^2} \frac{\partial \Phi}{\partial r} + \frac{q^2}{pr^4} (1 + 2\beta), \end{aligned} \quad (5)$$

$$(R_{rr} = 8\pi T_{rr}):$$

$$\frac{\partial^2}{\partial r^2} (\alpha + \gamma + \delta) - \frac{2}{r} \frac{\partial}{\partial r} (\beta - \gamma - \delta) + \frac{p'}{2p} \frac{\partial}{\partial r} (3\alpha - \beta + \gamma + \delta)$$

$$\begin{aligned}
& + \frac{1}{pr^2} \left(\frac{\partial^2 \beta}{\partial \theta^2} + \cot \theta \frac{\partial \beta}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \beta}{\partial \varphi^2} \right) \\
& = \frac{2q}{pr^2} \frac{\partial \Phi}{\partial r} - \frac{q^2}{pr^4} (1 + 2\alpha), \tag{6}
\end{aligned}$$

$(R_{\theta\theta} = 8\pi T_{\theta\theta})$:

$$\begin{aligned}
& \frac{\partial^2 \gamma}{\partial r^2} + \frac{p'}{p} \frac{\partial \gamma}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (\alpha - \beta + 3\gamma + \delta) - \frac{2}{pr^2} (\beta - \gamma) \\
& + \frac{1}{pr^2} \left[\frac{\partial^2}{\partial \theta^2} (\alpha + \beta + \delta) + \cot \theta \frac{\partial}{\partial \theta} (2\delta - \gamma) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \gamma}{\partial \varphi^2} \right] \\
& = \frac{2q}{pr^2} \left(\frac{\partial \Phi}{\partial r} + \frac{q}{r^2} \alpha \right), \tag{7}
\end{aligned}$$

$(R_{\varphi\varphi} = 8\pi T_{\varphi\varphi})$:

$$\begin{aligned}
& \frac{\partial^2 \delta}{\partial r^2} + \frac{p'}{p} \frac{\partial \delta}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (\alpha - \beta + \gamma + 3\delta) - \frac{2(\beta - \gamma)}{pr^2} \\
& + \frac{1}{pr^2} \left[\frac{\partial^2 \delta}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} (\alpha + \beta - \gamma + 2\delta) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} (\alpha + \beta + \gamma) \right] \\
& = \frac{2q}{pr^2} \left(\frac{\partial \Phi}{\partial r} + \frac{q}{r^2} \alpha \right), \tag{8}
\end{aligned}$$

$(R_{r\theta} = 8\pi T_{r\theta})$:

$$\frac{\partial^2}{\partial r \partial \theta} (\alpha + \delta) + \cot \theta \frac{\partial}{\partial r} (\delta - \gamma) + \frac{p'}{2p} \frac{\partial}{\partial \theta} (\alpha - \beta) - \frac{1}{r} \frac{\partial}{\partial \theta} (\alpha + \beta) = -\frac{2q}{pr^2} \frac{\partial \Phi}{\partial \theta}, \tag{9}$$

$(R_{r\phi} = 8\pi T_{r\phi})$:

$$\frac{\partial^2}{\partial r \partial \varphi} (\alpha + \gamma) + \frac{p'}{2p} \frac{\partial}{\partial \varphi} (\alpha - \beta) - \frac{1}{r} \frac{\partial}{\partial \varphi} (\alpha + \beta) = -\frac{2q}{pr^2} \frac{\partial \Phi}{\partial \varphi}, \tag{10}$$

$(R_{\theta\varphi} = 8\pi T_{\theta\varphi})$:

$$\frac{\partial^2}{\partial \theta \partial \varphi} (\alpha + \beta) - \cot \theta \frac{\partial}{\partial \varphi} (\alpha + \beta) = 0, \tag{11}$$

$(\nabla_a F^{ta} = 0)$:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{pr^2} \left(\frac{\partial^2 \Phi}{\partial \theta^2} + \cot \theta \frac{\partial \Phi}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} \right) + \frac{q}{r^2} \frac{\partial}{\partial r} (\alpha + \beta - \gamma - \delta) = 0. \tag{12}$$

Resolution

Since all static and axisymmetric solutions to the Einstein-Maxwell equations belong to Weyl's *electrovac* class [5], it is convenient to define the five perturbation functions in such a way as to avoid the repetition of these known results. On the other hand, it is also convenient to exclude the possibility that those perturbations arise simply as a consequence of an inappropriate choice of coordinate origin, in which case even the spherically symmetric solution would possess such non-zero metric functions. Both exclusions may be easily achieved in the selected coordinate system, by means of an expansion in spherical harmonics in the usual way:

$$\alpha(r, \theta, \varphi) = \sum_{n=2}^{\infty} \sum_{\substack{l=-n \\ (l \neq 0)}}^n \alpha_{nl}(r) Y_n^l(\theta, \varphi), \quad (13)$$

and other similar expansions for the remaining functions β , γ , δ and Φ . It is important to notice that the chosen set of indexes n, l explicitly imposes the above mentioned restrictions, for $n = 0$ represents the spherically symmetric terms, which are already fully present in the background Reissner-Nordström curvature; the $l = 0$ set corresponds to the axisymmetric multipoles; and finally, a non-vanishing $n = 1, l = \pm 1$ term simply means a shifted choice of coordinate origin.

Keeping this in mind, we notice that the field equations (11), (10) and (9) impose the following relations between the perturbation functions

$$\begin{aligned} \beta &= -\alpha, \\ \frac{\partial \gamma}{\partial r} &= - \left[\frac{2q}{pr^2} \Phi + \frac{\partial \alpha}{\partial r} + \frac{p'}{p} \alpha \right], \\ \delta &= \gamma; \end{aligned} \quad (14)$$

whose inclusion in the remaining field equations show that (5) and (6), as well as (7) and (8) become identical, leaving us with just three independent equations. Using the expansions (13), and adopting the notation $\tilde{\alpha}(r)$, $\tilde{\gamma}(r)$ and $\tilde{\Phi}(r)$ to represent the set of radial coefficients

α_{nl} , γ_{nl} and Φ_{nl} , all of them with the same index n , these equations reduce to a system of two ordinary differential equations on $\tilde{\alpha}$ and $\tilde{\Phi}$, namely:

$$\frac{d^2\tilde{\alpha}}{dr^2} + \left[\frac{p'}{p} + \frac{2}{r} \right] \frac{d\tilde{\alpha}}{dr} + \left[\frac{2q^2}{pr^4} - \frac{p'^2}{p^2} - \frac{n(n+1)}{pr^2} \right] \tilde{\alpha} = \frac{2q}{pr^2} \left[\frac{p'}{p} \tilde{\Phi} - \frac{d\tilde{\Phi}}{dr} \right], \quad (15a)$$

$$\frac{d^2\tilde{\Phi}}{dr^2} + \frac{2}{r} \frac{d\tilde{\Phi}}{dr} + \left[\frac{4q^2 - n(n+1)r^2}{pr^4} \right] \tilde{\Phi} = -\frac{2q}{r^2} \left[\frac{d\tilde{\alpha}}{dr} + \frac{p'}{p} \tilde{\alpha} \right], \quad (15b)$$

plus an algebraic relation for $\tilde{\gamma}$

$$\frac{d^2\tilde{\alpha}}{dr^2} + 2 \left[\frac{p'}{p} + \frac{1}{r} \right] \frac{d\tilde{\alpha}}{dr} + \frac{2}{r^2} \left[\frac{q^2}{pr^2} - 1 \right] \tilde{\alpha} + \frac{(n+2)(n-1)}{pr^2} \tilde{\gamma} = -\frac{4q}{pr^3} \left[\tilde{\Phi} + r \frac{d\tilde{\Phi}}{dr} \right]. \quad (16)$$

In this way, a simultaneous solution $[\tilde{\alpha}(r); \tilde{\Phi}(r)]$ of (15), would determine, via (16), (14) and the expansions (13), the Einstein-Maxwell fields (1) and (3).

The so obtained field equations show the convenience of the assumed linearization in a Reissner-Nordström background, since the absence of cross-terms has eliminated the l -dependence in (15) for each $(2n)$ -set of multipoles with the same index n , thus recovering the well-known fact, that it is always possible to restrict ourselves to an axisymmetric dependence, completely determined by the index n .

In the limiting case of null electric field, $q = \Phi = 0$, our set of independent equations (15) reduces only to one differential equation on the gravitational perturbation $\tilde{\alpha}(r)$, namely

$$\frac{d^2\tilde{\alpha}}{dr^2} + \frac{2(r-m)}{r(r-2m)} \frac{d\tilde{\alpha}}{dr} - \frac{4m^2 + n(n+1)(r-2m)r}{r^2(r-2m)^2} \tilde{\alpha} = 0. \quad (17)$$

The simple transformation from our radial curvature coordinate r to the radial isotropic coordinate ρ through a new dimensionless radial parameter x ,

$$x = \left(\frac{m}{2\rho} \right)^2 \rightarrow r = \frac{m}{2} \frac{1 + \sqrt{x}}{\sqrt{x}}, \quad (18)$$

converts (15a) into

$$(1-x)^2 x^2 \frac{d^2\tilde{\alpha}}{dx^2} + \frac{1}{2} (1-3x) (1-x) x \frac{d\tilde{\alpha}}{dx} - \left[4x + \frac{1}{4} n(n+1) (1-x)^2 \right] \tilde{\alpha} = 0, \quad (19)$$

for $\{0 \leq x < 1\}$ corresponding to the region $\{2m < r \leq \infty\}$, which is identical to the one obtained in [1]. Obviously, the form of (17), which is an homogeneous differential equation

with three regular singularities, ensures that it is possible to obtain exact solutions directly from it, but their demonstrated equivalence with the Bondi-Rindler solutions doesn't make it worth while.

III. ANALYSIS OF THE EQUATIONS

We would like to emphasize that the electric and gravitational fields are completely determined by (15), which are a coupled system of linear differential equations. The general relativistic coupling between these fields is obvious, since both equations possess on their right hand side an inhomogeneous term. Therefore, the general solution of (15) can be split into an homogeneous part for each function $\tilde{\alpha}$ and $\tilde{\Phi}$, plus the inhomogeneous, coupled terms.

Furthermore, (15a) and (15b) are elliptic equations, and hence they define a well-posed boundary problem, whose boundary conditions must be imposed as physically meaningful constraints. In addition, their elliptic character offers an extra advantage, for the required boundary conditions will have to be satisfied not only by the entire solutions $[\tilde{\alpha}(r); \tilde{\Phi}(r)]$, but also by their homogeneous parts $[\tilde{\alpha}_0(r); \tilde{\Phi}_0(r)]$, and this can be simply checked out by means of the simpler set of homogeneous equations, given by the left hand sides of (15).

The most obvious boundary condition was already assumed in the definition of the perturbations, for these must vanish at large distances from the distribution in order to recover the Reissner-Nordström solution. In this way, it is straightforward to verify that both $\tilde{\alpha}$ and $\tilde{\Phi}$ are determined by two arbitrary integration constants, and therefore, for both fields, each set of harmonics possesses the same multiplicity of solutions as the classical study of independent gravitational and electrostatic potentials. Likewise, in each case only one of the two possible solutions will vanish as they approach infinity. On the other hand, the second boundary condition should be imposed by the junction with a convenient interior solution; but its arbitrary character would make it very difficult to define in a tractable way. Fortunately, the metric (1) inherits a physical feature from the chosen background curvature,

at least for the general cases $q^2 \leq m^2$, since the analytic extension of the Reissner-Nordström solution describes a black hole, with an event horizon situated at $r_+ = m + \sqrt{m^2 - q^2}$, and its “no-hair” property brings a suitable boundary condition. This can be easily shown analytically, and without invalidating the generality of the above remark, if we restrict ourselves to the simpler case $q^2 = m^2$. The homogeneous parts of (15) can be written, in terms of a convenient dimensionless radial parameter $x = \frac{m}{r}$, as follows:

$$x^2 (1-x)^2 \frac{d^2 \tilde{\alpha}_0}{dx^2} + 2x^2 (x-1) \frac{d\tilde{\alpha}_0}{dx} - [2x^2 + n(n+1)] \tilde{\alpha}_0 = 0, \quad (20a)$$

$$x^2 (1-x)^2 \frac{d^2 \tilde{\Phi}_0}{dx^2} + [4x^2 - n(n+1)] \tilde{\Phi}_0 = 0. \quad (20b)$$

In this way, the $r \rightarrow \infty$ boundary condition is now located at the finite value $x = 0$, and the event horizon for the maximal charged black hole at $x = 1$, so that the outside space is covered by the region $\{0 < x \leq 1\}$. As both equations now possess three regular singularities at $x = 0, 1, \infty$, the former are granted to have solutions with an hypergeometric dependence given by some functions $h(x)$, defined as follows:

$$\tilde{\alpha}_0 = x^{p_1} (1-x)^{q_1} h_1(x) \quad , \quad \tilde{\Phi}_0 = x^{p_2} (1-x)^{q_2} h_2(x); \quad (21)$$

whose insertion in (20) brings up the following pairs of solutions for the exponents p, q , suitably expressed in terms of an index $j = n + \frac{1}{2}$:

$$p_1, p_2 = \begin{cases} n+1 \\ n \end{cases} \quad , \quad q_1 = -\frac{1}{2} \pm \sqrt{j^2 + 2} \quad , \quad q_2 = +\frac{1}{2} \pm \sqrt{j^2 - 4}. \quad (22)$$

It is then obvious that only the solutions with p equal to $n + 1$ will satisfy the asymptotic vanishing condition. On the other hand, the q 's with positive root must be discarded, for they would lead to vanishing perturbations at the event horizon $x = 1$, showing a spherically symmetric configuration at the finite radial value $r = m$; this situation would be in flagrant contradiction with Birkhoff's theorem. In this way, we are led to a unique set of choices; with these, (20) can be cast into the canonical form of standard hypergeometric equations

$$x(1-x) \frac{d^2 h_1}{dx^2} + \left[2(n+1) - x \left(2n+3 + 2\sqrt{j^2+2} \right) \right] \frac{dh_1}{dx}$$

$$+ \left[2(n+1)\sqrt{j^2+2} - 2j(n+1) \right] h_1 = 0, \quad (23a)$$

$$x(1-x)\frac{d^2 h_2}{dx^2} + \left[2(n+1) - x \left(2n+3 + 2\sqrt{j^2-4} \right) \right] \frac{dh_2}{dx} + \left[2(n+1)\sqrt{j^2-4} - 2j(n+1) \right] h_2 = 0; \quad (23b)$$

whose integrals are given by the hypergeometric functions

$$h_1(x) = H(a_1, b_1, 2n+2; x), \quad h_2(x) = H(a_2, b_2, 2n+2; x); \quad (24)$$

with the a 's and b 's defined according to

$$\begin{cases} a_1 + b_1 = 2(n+1 - \sqrt{j^2+2}) \\ a_1 b_1 = 2(n+1)(j - \sqrt{j^2+2}) \end{cases}, \quad \begin{cases} a_2 + b_2 = 2(n+1 - \sqrt{j^2-4}) \\ a_2 b_2 = 2(n+1)(j - \sqrt{j^2-4}) \end{cases}. \quad (25)$$

Although in both cases a and b are complex numbers, the regularity of h_1 and h_2 is completely ensured by their hypergeometric definition

$$H(a, b, c; x) = 1 + \frac{ab}{c} \frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad (26)$$

since all a and b solutions of (25) are complex conjugates, the functions in (26) are always real-valued. It is then obvious that h_1 and h_2 possess a regular behaviour in the $[0; 1]$ range; moreover, their hypergeometric dependence ensures also that they are monotonously increasing functions in that same range, as can be easily seen by keeping in mind their vanishing at $x = 0$, and evaluating them and their first derivatives at $x = 1$. So, the radial dependence of $\tilde{\alpha}$ and $\tilde{\Phi}$ will be mostly dominated by the $x^p(1-x)^q$ terms in (21): For $x \rightarrow 0$, (22) shows that the asymptotic behaviour will be determined by the x^{n+1} term, thus inducing a radial dependence given by

$$\tilde{\alpha}, \tilde{\Phi} \sim \frac{1}{r^{n+1}}, \quad (27)$$

which corresponds to the classical coulombian decreasing rate; this also shows that the coupling terms between both fields, i.e. those who are non-coulombian contributions, asymptotically vanish with greater powers of the radial coordinate, and only the harmonic (27)

part of the fields will be perceptible. On the other side, both $\tilde{\alpha}$ and $\tilde{\Phi}$ diverge in the $x \rightarrow 1$ limit, since their radial dependence will be dominated by the $(1-x)^q$ terms, and the q 's defined in (22) are all negative for the permitted values of n . This divergent boundary condition at $x = 1$ is in agreement with the “no-hair” theorem for charged black holes, since it ensures that there can be no bounded static perturbations of the analytically extended electrovac solutions on the event horizon. It also shows that in the $x \sim 1$ strong-field region, the coupling between both fields dominates over their coulombian parts, given by (27). The homogeneous functions $[\tilde{\alpha}_0; \tilde{\Phi}_0]$ so obtained also offer an algorithmic way to compute the complete inhomogeneous field equations, since these functions can now be substituted into the right hand sides of (15) in order to find some first inhomogeneous functions $[\tilde{\alpha}_i; \tilde{\Phi}_i]$. Iterating this procedure leads to more accurate inhomogeneous functions, which, when added to the homogeneous ones, will bring exact solutions to the field equations. Anyway, it must be born in mind that these functions might not be exact solutions of the field equations out of the weak-field condition, where their smallness condition has to be relaxed. Nevertheless, as discussed in [1], the ellipticity of (15) ensures that no solutions other than these can exist outside of a sphere which contains all the sources.

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