Conformal symmetry inheritance in null fluid spacetimes

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Abstract
We define inheriting conformal Killing vectors for null fluid spacetimes and
find the maximum dimension of the associated inheriting Lie algebra. We show
that for non-conformally flat null fluid spacetimes, the maximum dimension
of the inheriting algebra is seven and for conformally flat null fluid spacetimes
the maximum dimension is eight. In addition, it is shown that there are two
distinct classes of non-conformally flat generalized plane wave spacetimes
which possess the maximum dimension, and one class in the conformally flat
case.

PACS numbers: 04.20.Jb, 04.40.Nr, 02.40.Ky

1. Conformal symmetry inheritance

Knowledge of the symmetry group of a spacetime is a useful tool in constructing spacetime
solutions of Einstein's field equations and also in classifying known solutions according to the
Lie algebra structure generated by these symmetries. While much is known about isometries
and homotheties, comparatively little is known about conformal symmetries. This is, at least
in part, due to the difficulty in solving the conformal Killing equations which contain the
conformal scalar ψ, in general a function of all four coordinates. Some slight simplification
of these equations occurs in the case of perfect fluid spacetimes if one searches for inheriting
conformal Killing vectors [1–5], and this simplification has led to the discovery of all spherical
and plane symmetric perfect fluid spacetimes admitting inheriting conformal Killing vectors.
One result of particular interest is that the maximum dimension of the Lie algebra of inheriting
conformal Killing vectors is eight for conformally flat perfect fluid spacetimes [6] and five for
non-conformally flat perfect fluid spacetimes [7].

In this paper, we extend the concept of inheritance to null fluid (pure radiation) spacetimes. We define inheriting conformal Killing vector fields for such spacetimes and determine the maximum possible dimension of the Lie algebra of inheriting conformal Killing vectors for conformally flat and for non-conformally flat spacetimes. As a result of this study, we are able to clarify and extend some of the previous work on the symmetries of generalized plane wave spacetimes [8, 9], and to provide some insight into the conformal relations between members of this class of spacetimes.

Let M be a four-dimensional spacetime manifold with metric tensor **g** of Lorentz signature. Any vector field ξ which satisfies

$$\mathcal{L}_\xi \mathbf{g} = 2\psi(x^\alpha)\mathbf{g} \tag{1}$$

is said to be a *conformal Killing vector* (CKV) of **g**. If ψ is not constant on M then ξ is called a *proper conformal Killing vector*, if $\psi_{;ab} = 0$ then ξ is called a *special conformal Killing vector* (SCKV), if ψ is constant on M then ξ is called a *homothetic Killing vector* (HKV) and if $\psi = 0$ then ξ is said to be a Killing vector (KV). An SCKV is called a proper SCKV if $\psi_{;a} \neq 0$ and an HKV is called a proper HKV if $\psi \neq 0$. The set of all CKV (respectively, SCKV, HKV and KV) forms a finite-dimensional Lie algebra denoted by $\mathcal{C}$ (respectively, $\mathcal{S}$, $\mathcal{H}$ and $\mathcal{G}$). The maximum dimension of the algebra of CKV on M is 15 and this is achieved if M is conformally flat. If the spacetime is not conformally flat, then the maximum dimension is seven, and this occurs for special type N solutions. Indeed, if the Petrov type of a spacetime (M, g) is N, $\dim \mathcal{C} \leqslant 7$ [10]. It can be shown that [11]

$$\mathcal{L}_\xi R_{ab} = -2\psi_{;ab} - g_{ab}\Box\psi, \tag{2}$$

$$\mathcal{L}_\xi R = -2\psi R - 6\Box\psi, \tag{3}$$

where $\Box\psi = g^{ab}\psi_{;ab}$.

Let us first remark on the definition of inheritance. Consider a spacetime admitting one or more KV and some geometrical object X (scalar, vector, tensor, etc) defined on this spacetime. We say that X inherits the symmetry of the KV, ξ, if $\mathcal{L}_\xi X = 0$. We are interested in those objects that inherit the symmetries of all members of $\mathcal{G}$. If we consider the case of a *perfect fluid* spacetime, we find that the timelike fluid velocity vector V naturally inherits the symmetry of any KV of the spacetime, i.e., $\mathcal{L}_\xi V = 0$ for all $\xi \in \mathcal{G}$. Thus, V is the natural object of interest in the context of inheritance. If the spacetime admits an HKV, ξ, the field equations imply that [2]

$$\mathcal{L}_\xi V^a = -\psi V^a, \qquad \mathcal{L}_\xi V_a = \psi V_a. \tag{4}$$

This is the definition of inheritance of a homothetic symmetry in a perfect fluid spacetime. This definition of inheritance for a perfect fluid spacetime is also supported by dimensional considerations [12]. Equations (4) also hold for SCKV but, in the context of perfect fluid spacetimes, this is of no consequence since no perfect fluid spacetime can admit an SCKV [2].

If the spacetime admits a CKV it does not follow, in general, that the Lie derivative of V with respect to the CKV will be parallel to V, but if the relations (4) do hold for the CKV then it is said to be an inheriting CKV (ICKV) with respect to the fluid flow vector V. We note that the set of ICKV forms a subalgebra $\mathcal{I}$ of the Lie algebra $\mathcal{C}$ (this is proved in [13]) and we refer to this as the inheriting algebra.

Let us now turn to the case of a null fluid. The field equations are (cosmological constant $\Lambda = 0$)

$$G_{ab} = R_{ab} = Fk_ak_b, \tag{5}$$

where the function $F > 0$ for energy conditions to hold and k is null. If ξ is a KV, the null vector k satisfies $\mathcal{L}_\xi k_a = \alpha k_a$ where α is a scalar function of the coordinates which, in general,

is not zero with respect to every KV of $\mathcal{G}$. Thus, k_a is not the required object that inherits the symmetries of $\mathcal{G}$. However, if we define

$$V_a = c\sqrt{F}k_a, \tag{6}$$

where c is a non-zero constant, the field equations become

$$R_{ab} = c^{-2}V_aV_b \tag{7}$$

and, since (2) implies $\mathcal{L}_\xi R_{ab} = 0$, it follows that $\mathcal{L}_\xi V_a = 0$ for all KV of $\mathcal{G}$, so that V_a is the inheriting object analogous to the fluid velocity vector of the perfect fluid case. We may regard V_a as the null fluid velocity vector. Note that in equations (6) and (7), without loss of generality, we may put $c = 1$, which henceforth we shall do.

For an HKV ξ, (2) again gives $\mathcal{L}_\xi R_{ab} = 0$ and, from equation (7) it follows that for any HKV the inheriting vector satisfies

$$\mathcal{L}_\xi V^a = -2\psi V^a, \qquad \mathcal{L}_\xi V_a = 0. \tag{8}$$

Following in the spirit of the definition of ICKV in the perfect fluid case, we define ICKV for a null fluid as those CKV (including automatically KV and HKV) that satisfy (8), or equivalently $\mathcal{L}_\xi R_{ab} = 0$. Since $R = V^aV_a = 0$, (3) implies $\Box\psi = 0$ and hence (2) implies that for a CKV $\mathcal{L}_\xi R_{ab} = -2\psi_{;ab}$, so that every ICKV must be a SCKV. Thus, the ICKV in the null fluid case consist of the set of all SCKV of the spacetime, and so, form a finite-dimensional subalgebra $\mathcal{I}$ of the Lie algebra $\mathcal{C}$. From now on (unless indicated otherwise) when referring to an ICKV, we shall mean a CKV satisfying relations (8).

We determine the maximum dimension of $\mathcal{I}$ in section 2. For the non-conformally flat spacetimes, the maximum possible dimension of the CKV algebra occurs when the spacetime is conformal to a (special) generalized plane wave spacetime [10]. There are only two classes of conformally flat null fluid spacetime, the (special) generalized plane wave spacetimes and the Edgar–Ludwig spacetimes [14, 15]. We note that, if a non-conformally flat generalized plane wave spacetime admits a $\mathcal{C}_7$ it need not be an $\mathcal{S}_7$ [16]. In section 3, we consider certain conformal factors which preserve the maximum dimension.

2. The maximum dimension of the inheriting algebra

2.1. Inheritance in generalized plane wave spacetimes

The details of the conformal symmetry properties of the null fluid pp-wave spacetimes are given in [8, 9, 16] and [17]. The line element for such a pp-wave spacetime can be written [18] as

$$ds^2 = -2du\,dv - 2H(u, y, z)\,du^2 + dy^2 + dz^2. \tag{9}$$

The null covariantly constant vector k is necessarily a KV and has the form

$$k^a = \delta_v^a, \qquad k_a = -\delta_a^u. \tag{10}$$

In this case, $F = H_{,yy} + H_{,zz}$. In the special case of a null Einstein–Maxwell generalized plane wave spacetime, the function H can be put in the form

$$2H = A(u)y^2 + 2B(u)yz + C(u)z^2 \tag{11}$$

and (9) admits at least an $\mathcal{H}_6$. When $A(u) = -C(u)$ the spacetime is vacuum and when $A(u) = C(u)$ and $B(u) = 0$, the spacetime is conformally flat and admits an $\mathcal{H}_7$ subalgebra.

We wish to determine the maximum number of ICKV admitted by the spacetimes of the form (9) with H given by (11). The KVs k, X_i and HKV Z are a basis for the Lie algebra $\mathcal{H}_6$, i.e.,

$$X_i = d_i(u)\frac{\partial}{\partial y} + e_i(u)\frac{\partial}{\partial z} + (yd_i'(u) + ze_i'(u))\frac{\partial}{\partial v}, \tag{12}$$

where the prime denotes differentiation with respect to u, $i = 1, \ldots, 4$ and there are four sets of functions $(d_i(u), e_i(u))$ each of which satisfies

$$d_i(u)C(u) + e_i(u)B(u) + d_i''(u) = 0, \tag{13}$$

$$d_i(u)B(u) + e_i(u)A(u) + e_i''(u) = 0, \tag{14}$$

and

$$Z = 2v\frac{\partial}{\partial v} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{15}$$

Thus, all generalized plane wave spacetimes admit at least an $\mathcal{I}_6$.

2.1.1. Non-conformally flat spacetimes.

For non-conformally flat spacetimes there are specializations of the functions $A(u)$, $B(u)$, $C(u)$ in (11) which give rise to one additional symmetry satisfying the SCKV condition, giving an $\mathcal{S}_7$. Since $\dim \mathcal{C} \leqslant 7$ for any non-conformally flat spacetime, it follows that the maximum dimension of the inheriting algebra for non-conformally flat spacetimes is seven. There are six such specializations, four of these, namely cases 11–14 of table 2 in [8] give rise to an additional KV, and two give rise to a proper SCKV. We list the six types.

(i) Case 11 of [8]. The function H has the form

$$A(u) = au^{-2}, \qquad B(u) = bu^{-2}, \qquad C(u) = cu^{-2}, \tag{16}$$

where a, b and c are constants. The additional KV is

$$X_6 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{17}$$

(ii) Case 12 of [8]. The function H has the form

$$A(u) = cu^{-2}(\sin\phi + l), \qquad B(u) = cu^{-2}\cos\phi, \qquad C(u) = cu^{-2}(-\sin\phi + l), \tag{18}$$

where $\phi = 2\gamma \ln|u|$, and c, l and γ are constants. The additional KV is

$$X_6 = \gamma\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) + u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{19}$$

(iii) Case 13 of [8]. The function H has the form

$$A(u) = a, \qquad B(u) = b, \qquad C(u) = c, \tag{20}$$

where a, b and c are constants. The additional KV is

$$X_6 = \frac{\partial}{\partial u}. \tag{21}$$

(iv) Case 14 of [8]. The function H has the form

$$A(u) = c\sin\phi + l, \qquad B(u) = c\cos\phi, \qquad C(u) = -c\sin\phi + l, \tag{22}$$

where $\phi = 2\gamma u$, and c, l and γ are constants. The additional KV is

$$X_6 = \frac{\partial}{\partial u} + \gamma\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right). \tag{23}$$

(v) The function H has the form

$$A(u) = c(u^2 + \beta)^{-2}(\sin\phi + l), \qquad B(u) = c(u^2 + \beta)^{-2}\cos\phi,$$
$$C(u) = c(u^2 + \beta)^{-2}(-\sin\phi + l), \qquad (24)$$

where

$$\phi = 2\gamma \int (u^2 + \beta)^{-1}\,du \qquad (25)$$

and c, β and γ are constants. The proper SCKV is

$$W = (u^2 + \beta)\frac{\partial}{\partial u} + \frac{1}{2}(y^2 + z^2)\frac{\partial}{\partial v} + u\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) + \gamma\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \qquad (26)$$

with conformal scalar $\psi = u$. We emphasize that this is the form for a *proper* SCKV and it is worth noting that the form for a general (i.e., not necessarily proper) SCKV for a generalized plane wave spacetime can be written in a slightly different form; see [16].

(vi) The function H has the form

$$A(u) = -a(u^2 + \beta)^{-2}, \qquad B(u) = -b(u^2 + \beta)^{-2}, \qquad C(u) = -c(u^2 + \beta)^{-2}, \qquad (27)$$

where a, b and c are constants. The corresponding proper SCKV is

$$W = (u^2 + \beta)\frac{\partial}{\partial u} + \frac{1}{2}(y^2 + z^2)\frac{\partial}{\partial v} + u\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right). \qquad (28)$$

Setting $\gamma = 0$ in (ii) and (iv) leads to classes of functions of H more restricted than permitted in (i) and (iii). Similarly, setting $\gamma = 0$ in (v) leads to classes of functions of H more restricted than permitted in (vi).

2.1.2. Conformally flat spacetimes.
The conformally flat generalized plane wave spacetimes have metrics of the form

$$ds^2 = -A(u)(y^2 + z^2)\,du^2 - 2du\,dv + dy^2 + dz^2. \qquad (29)$$

Conformally flat spacetimes admit a $\mathcal{C}_{15}$. Such spacetimes admit an $\mathcal{H}_7$, with basis k, X_i, Z given by equations (10), (12) and (15), respectively and another KV given by

$$X_7 = z\frac{\partial}{\partial z} - y\frac{\partial}{\partial y}. \qquad (30)$$

Thus, spacetimes of the form (29) admit at least an $\mathcal{I}_7$. Since conformally flat spacetimes admit a $\mathcal{C}_{15}$, it follows that there are in general eight CKV which are all proper. However, there are three specializations of the function $A(u)$ for which one of the proper CKV becomes an SCKV. Thus, the maximum dimension of the inheriting algebra for conformally flat generalized plane wave spacetimes is eight.

(vii) Case 16 of [8]. The function H has the form

$$A(u) = a, \qquad (31)$$

where a is a constant. The additional KV is

$$X_6 = \frac{\partial}{\partial u}. \qquad (32)$$

(viii) Case 17 of [8]. The function H has the form

$$A(u) = au^{-2}, \qquad (33)$$

where a is a constant. The additional KV is

$$X_6 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \qquad (34)$$

(ix) The third specialization occurs when H has the form

$$A(u) = a(u^2 + \beta)^{-2},\tag{35}$$

where a and β are constants. The corresponding symmetry is a proper SCKV W given by equation (28).

Thus, the three classes of conformally flat spacetimes given by (29) with (31), (33), (35) each admit eight ICKV and the maximum dimension of the inheriting algebra is eight.

2.2. Inheritance in Edgar–Ludwig spacetimes

The most general conformally flat pure radiation spacetime which is *not* a pp-wave spacetime is given by [14, 15]

$$ds^2 = (xV(u, x, y) - w^2)\,du^2 + 2x\,du\,dw - 2w\,du\,dx - dx^2 - dy^2\tag{36}$$

where

$$V = x^2 + y^2 + 2M(u)x + 2F(u)y + 2S(u)\tag{37}$$

and M, F and S are arbitrary functions of the coordinate u. Since (36) is conformally flat, it admits a $\mathcal{C}_{15}$. Barnes [19] gives the general solution to the CKV equation (1) for the metric (36) in terms of the functions M, F and S and 15 unknown functions of u, i.e., in the notation of [19], a, b, c and their first and second derivatives, d and its first derivative and β, γ, ϵ and k. Barnes has shown that the maximum possible dimension of the homothety algebra for the spacetime (36) is two and the maximum possible dimension of the isometry algebra for this spacetime is one.

It is straightforward to show that for a CKV to be an SCKV $a = b = c = \beta = 0$, ϵ and γ are constants and the conformal scalar must have the form

$$\psi = \gamma.$$

It follows that the only SCKV admitted by the metric (36) are the HKV and KV determined by Barnes. Thus, the maximum possible number of ICKV admitted by this metric is two and the metric will admit no proper ICKV.

3. Conformally related spacetimes

Given any null fluid spacetime with line element ds^2 and Ricci tensor given by (5), then the conformally related spacetime

$$d\bar{s}^2 = \Omega^2(u)\,ds^2\tag{38}$$

corresponds to another null fluid with Ricci tensor

$$\bar{R}_{ab} = \bar{V}_a\bar{V}_b\tag{39}$$

where

$$\bar{V}_a = \sqrt{\bar{F}}k_a = [F - 2(\ln\Omega)_{,uu} + 2(\ln\Omega)_{,u}(\ln\Omega)_{,u}]^{\frac{1}{2}}k_a.\tag{40}$$

This is straightforward to verify. The Ricci tensors of the conformally related metrics are related by [20]

$$\bar{R}_{ab} = R_{ab} - 2\nabla_a\nabla_b\ln\Omega + 2(\nabla_a\ln\Omega)(\nabla_b\ln\Omega)$$
$$- g_{ab}g^{cd}\nabla_c\nabla_d\ln\Omega - 2g_{ab}g^{cd}(\nabla_c\ln\Omega)(\nabla_d\ln\Omega).\tag{41}$$

Now if we consider $\Omega = \Omega(u)$ only, then it immediately follows that

$$\nabla_a(\ln \Omega) = (\ln \Omega)_{,u} k_a$$
$$\nabla_a \nabla_b(\ln \Omega) = (\ln \Omega)_{,uu} k_a k_b,$$
(42)

the second equation resulting from the fact that k_a is a covariantly constant null vector. The second and third terms on the RHS of (41) are thus proportional to $k_a k_b$ and the last two terms in equation (41) vanish since they are proportional to the magnitude of the null vector k.

It is known that every generalized plane wave spacetime is conformally related to a vacuum plane wave spacetime [21]. Here we show that every generalized plane wave spacetime is conformally related to another generalized plane wave spacetime by a non-constant conformal factor. We define a new variable U by

$$\mathrm{d}U = \Omega^2(u)\,\mathrm{d}u$$
(43)

and write $\bar{\Omega}(U)$, $\bar{A}(U)$, $\bar{B}(U)$, $\bar{C}(U)$, for the functions $\Omega(u)$, $A(u)$, $B(u)$, $C(u)$, respectively, expressed as functions of U according to (43). Then the discrete transformation

$$u = \int (\bar{\Omega}(U))^{-2}\,\mathrm{d}U, \qquad v = V - (\ln \bar{\Omega}(U))_{,U}(Y^2 + Z^2)/2,$$
$$y = Y/\bar{\Omega}(U), \qquad z = Z/\bar{\Omega}(U),$$
(44)

allows us to put the metric (38) into the form of a generalized plane wave spacetime

$$\mathrm{d}\bar{s}^2 = -2K(U, Y, Z)\,\mathrm{d}U^2 - 2\mathrm{d}U\,\mathrm{d}V + \mathrm{d}Y^2 + \mathrm{d}Z^2,$$
(45)

where

$$2K(U, Y, Z) = [\bar{A}(U)\bar{\Omega}^{-4}(U) - \bar{\Omega}^{-1}(U)\bar{\Omega}_{,UU}(U)]Y^2 + 2\bar{B}(U)\bar{\Omega}^{-4}(U)YZ$$
$$+ [\bar{C}(U)\bar{\Omega}^{-4}(U) - \bar{\Omega}^{-1}(U)\bar{\Omega}_{,UU}(U)]Z^2.$$
(46)

Now, a CKV in a spacetime will be a CKV for any conformally related spacetime: however, the form of the conformal scalar ψ may not be preserved. The conformal factors are related by

$$\bar{\psi} = \xi^a(\ln \Omega)_{,a} + \psi.$$
(47)

Let us consider the restrictions on the form of $\Omega(u)$ in order that the maximum dimension r of the inheriting algebra $\mathcal{I}$ be maintained by the new spacetime (38). From the form of the five KV k, X_i, $i = 1, \ldots, 4$ and the HKV Z one can see immediately that they preserve the form of the conformal scalars, i.e., $\bar{\psi} = \psi$ for any functional form of $\Omega(u)$.

The only exceptions for the case of non-conformally flat spacetimes are the KV X_6 and the SCKV W which require special forms for $\Omega(u)$ in order to preserve the dimension of $\mathcal{I}$. Note that even though the dimension of $\mathcal{I}$ may be preserved, the new conformal scalar may not be so, e.g., $\bar{\psi} \neq \psi$.

We now use equations (43)–(46) to show that the SCKV spacetimes (v), (vi) and (ix) are not conformally related to each other, but each is conformally related to two of the non-SCKV spacetimes given by (ii) and (iv), (i) and (iii), (vii) and (viii), respectively.

3.1. Conformal class I

Consider the metric conformally related to (24), i.e.,

$$\mathrm{d}\bar{s}^2 = \Omega^2(u)[-c(u^2 + \beta)^{-2}((\sin \phi + l)y^2 + 2\cos \phi yz$$
$$+ (-\sin \phi + l)z^2)\,\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}v + \mathrm{d}y^2 + \mathrm{d}z^2].$$
(48)

(a) We first look for $\Omega(u)$ such that the SCKV (26) becomes a KV of (48). Since $\psi = u$ for the SCKV (26), equation (47) becomes

$$u + (\ln \Omega(u))_{,u}(u^2 + \beta) = 0$$

and it follows that

$$\Omega^2(u) = 1/(u^2 + \beta). \qquad (49)$$

Equation (43) gives

$$U = \int (u^2 + \beta)^{-1} \, \mathrm{d}u \qquad (50)$$

and hence, from (25),

$$\phi = 2\gamma U. \qquad (51)$$

Equations (44) put the metric (48) in the form

$$\mathrm{d}\bar{s}^2 = -[(c \sin \phi + \bar{l})Y^2 + 2c \cos \phi \, YZ + (-c \sin \phi + \bar{l})Z^2] \, \mathrm{d}U^2 - 2\mathrm{d}U \, \mathrm{d}V + \mathrm{d}Y^2 + \mathrm{d}Z^2, \qquad (52)$$

with $\bar{l} = cl + \beta$. Furthermore, we find that the SCKV (26) becomes the KV

$$X_6 = \frac{\partial}{\partial U} + \gamma \left(Z \frac{\partial}{\partial Y} - Y \frac{\partial}{\partial Z} \right).$$

Thus, the resulting spacetime is that of case (iv).

(b) We now look for $\Omega(u)$ such that the SCKV (26) becomes an HKV of (48). This HKV must be a linear combination of the HKV Z and the appropriate KV X_6. In this case equation (47) becomes

$$u + (\ln \Omega(u))_{,u}(u^2 + \beta) = \psi,$$

that is,

$$\Omega^2(u) = (u^2 + \beta)^{-1} \, \mathrm{e}^{2\psi\omega}, \qquad (53)$$

where

$$\omega = \int (u^2 + \beta)^{-1} \, \mathrm{d}u. \qquad (54)$$

Equation (43) now leads to

$$2\psi U = \mathrm{e}^{2\psi\omega}, \qquad (55)$$

and we find that the metric (48) is of the form

$$\mathrm{d}\bar{s}^2 = -\bar{c}U^{-2}[(\sin \phi + \bar{l})Y^2 + 2 \cos \phi \, YZ + (-\sin \phi + \bar{l})Z^2] \, \mathrm{d}U^2$$
$$- 2\mathrm{d}U \, \mathrm{d}V + \mathrm{d}Y^2 + \mathrm{d}Z^2, \qquad (56)$$

where $\bar{c} = \psi^{-2}c/4$ and $\bar{l} = l + (\psi^2 + \beta)/c$. After the rescaling $2\psi U = \bar{U}, 2\psi V = \bar{V}$, and defining $\bar{\gamma} = \gamma/2\psi$, the metric (56) is unchanged in form and, from equations (25), (54) and (55), we find $\phi = 2\gamma\omega = \gamma\psi^{-1} \ln |2\psi U|$, i.e.,

$$\phi = 2\bar{\gamma} \ln |\bar{U}|. \qquad (57)$$

The SCKV becomes the HKV

$$\xi = 2\psi\bar{U} \frac{\partial}{\partial \bar{U}} + \psi(Y + 2\bar{\gamma}Z) \frac{\partial}{\partial Y} + \psi(Z - 2\bar{\gamma}Y) \frac{\partial}{\partial Z}$$

which is the linear combination $\xi = 2\psi X_6 + \psi Z$. Thus, the resulting spacetime is that of case (ii).

3.2. Conformal class II

Consider the metric conformally related to (27), i.e.,

$$d\bar{s}^2 = \Omega^2(u)[-(u^2+\beta)^{-2}(ay^2+2byz+cz^2)\,du^2 - 2du\,dv + dy^2 + dz^2]. \tag{58}$$

(a) We find that the conformal factor (49) transforms the metric (58) into the form

$$d\bar{s}^2 = -[(a+\beta)Y^2 + 2bYZ + (c+\beta)Z^2]\,dU^2 - 2dU\,dV + dY^2 + dZ^2, \tag{59}$$

and the SCKV (28) becomes the KV

$$X_6 = \frac{\partial}{\partial U}. \tag{60}$$

Thus, the resulting spacetime is that of case (iii).

(b) We find that the conformal factor (53) transforms the metric (58) into the form

$$d\bar{s}^2 = -U^{-2}[\bar{a}Y^2 + 2\bar{b}YZ + \bar{c}Z^2]\,dU^2 - 2dU\,dV + dY^2 + dZ^2, \tag{61}$$

where $\bar{a} = (a+\beta+\psi^2)/4\psi^2$, $\bar{b} = b/4\psi^2$, $\bar{c} = (c+\beta+\psi^2)/4\psi^2$ and the SCKV (28) transforms into the linear combination $\xi = 2\psi X_6 + \psi Z$, where

$$X_6 = U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V}. $$

Thus, the resulting spacetime is that of case (i).

Let us now consider the type of conformal factor which will preserve the SCKV. We find that for both spacetimes (24) and (27) the conformal factor is

$$\Omega^2(u) = (U^2+\alpha)/(u^2+\beta) \tag{62}$$

where α is a constant. From equation (43), the coordinates are related by

$$\int (U^2+\alpha)^{-1}\,dU = \int (u^2+\beta)^{-1}\,du. \tag{63}$$

Multiplying the metric (24) with this $\Omega^2(u)$ and transforming according to equation (44), we obtain the metric coefficients

$$A(U) = c(U^2+\alpha)^{-2}(\sin\phi + \bar{l}), \qquad B(U) = c(U^2+\alpha)^{-2}\cos\phi, \tag{64}$$

$$C(U) = c(U^2+\alpha)^{-2}(-\sin\phi + \bar{l}), \tag{65}$$

where $c\bar{l} = cl - \alpha + \beta$. Equation (63) shows that the 'angle' ϕ, defined by equation (25), is unchanged. Hence, the resulting spacetime is of the same class as that in (24). Likewise, the metric (27) transforms into a metric of the same class as (27). Thus, spacetimes with metrics (24) and (27) are *not* conformally related to one another, so we have two sets of non-conformally flat spacetimes which admit the maximum number of seven ICKV. Each set consists of a class of SCKV spacetimes, (24) or (27), and two further classes of maximal ICKV spacetimes, conformally related to (24) or (27), in each of which the SCKV is replaced by a KV.

3.3. Conformal class III

Consider the conformally flat case, i.e., the SCKV spacetime (35).

(a) The conformal factor (49) transforms the metric (35) into the metric (31) of case (vii), i.e.,

$$d\bar{s}^2 = -\bar{a}(Y^2 + Z^2)\,dU^2 - 2dU\,dV + dY^2 + dZ^2, \tag{66}$$

where $\bar{a} = a + \beta$, and the SCKV (28) becomes the KV

$$X_6 = \frac{\partial}{\partial U}.$$

(b) The conformal factor (53) transforms the metric (35) into the metric (33) of case (viii), i.e.,

$$d\bar{s}^2 = -\bar{a}U^{-4}(Y^2 + Z^2)\,dU^2 - 2dU\,dV + dY^2 + dZ^2, \tag{67}$$

where $\bar{a} = (a + \beta + \psi^2)/4\psi^2$, and the SCKV (28) becomes the KV

$$X_6 = U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V},$$

with an added multiple of the HKV Z.

Thus, the null fluid ICKV spacetimes of maximal dimension are contained in the three sets of conformally related spacetimes I–III above. One of the sets consists of conformally flat spacetimes, each of which admits eight ICKV, while the other two sets consist of non-conformally flat spacetimes which are not conformally related, and admit seven ICKV. In each set there are three classes of maximal ICKV spacetimes which are conformally related by means of the conformal factors in (49), (53) and (62).

4. Discussion

In this paper, we have defined conformal symmetry inheritance for null fluid spacetimes and found the maximum dimension of the associated inheriting Lie algebra.

We have shown that for non-conformally flat spacetimes, the maximum dimension of the ICKV algebra is seven and this occurs when the metric is conformally related to a generalized plane wave spacetime with an $\mathcal{H}_7$. There are only two classes of conformally flat null fluid spacetimes, the generalized plane wave spacetimes and the Edgar–Ludwig spacetimes. In the former the maximum dimension of the ICKV algebra is eight, and in the latter the dimension is two. Thus, for conformally flat null fluid spacetimes the maximum dimension is eight. For a spacetime which is conformally related to a generalized plane wave spacetime, it is possible to restrict the form of the conformal factor $\Omega(u)$ in order to preserve the maximum dimension of the inheriting algebra. We have the following theorem.

Theorem. *For non-conformally flat null fluid spacetimes* $\dim \mathcal{I}$ *is at most seven, in which case the spacetime is conformally related to a type N generalized plane wave spacetime. For conformally flat null fluid spacetimes* $\dim \mathcal{I}$ *is at most eight in which case the spacetime is a type O generalized plane wave spacetime.*

There are no type N or O Robinson–Trautman pure radiation solutions [18], thus $\dim \mathcal{I}$ cannot be greater than six for this class of spacetimes. The general Vaidya spacetime admits a $\mathcal{G}_3$, and a special case will admit an $\mathcal{H}_4$ [22]. Hall and Carot [23] showed that the only type N Einstein–Maxwell spacetime to admit a proper CKV is a generalized plane wave spacetime and so, if searching for *proper* ICKV in type N Einstein–Maxwell spacetimes, attention must be limited to these solutions.

Acknowledgments

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