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Conformal symmetries in null Einstein–Maxwell fields

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Abstract. It is shown that if a null Einstein–Maxwell spacetime admits a proper conformal vector field it must either be a (generalized type N) pp-wave or a (generalized type III) Goldberg–Kerr metric.

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1. Introduction

There has been much recent interest in the study of conformal symmetries in spacetime. This paper is intended as a further contribution to this area of research and deals with the existence of proper conformal symmetries in null Einstein–Maxwell fields. It extends some recent work by Lewandowski (1990) in the case of vanishing cosmological constant and shows that, roughly speaking, if such a spacetime admits a proper conformal vector field then it is either of Petrov type N with a covariantly constant (null) ray vector or of Petrov type III with a properly recurrent ray vector. The first of these possibilities generalizes the (vacuum) pp-wave spacetimes (Ehlers and Kundt 1962) and the second generalizes the vacuum metrics discovered by Goldberg and Kerr (1962) and Kerr and Goldberg (1962). The result generalizes the known fact that the only vacuum spacetimes which can admit proper conformal vector fields are the pp-waves and which is a trivial consequence of Brinkmann's theorem (Brinkmann 1925, Ehlers and Kundt 1962, Hall 1983).

Throughout, M will denote a smooth spacetime manifold with smooth metric g of Lorentz signature. The corresponding Riemann, Ricci and Weyl tensors are denoted in component form by R_{abcd}, R_{ab} ($\equiv R^c{}_{acb}$) and C_{abcd}. In particular, (M, g) is assumed to be a null Einstein–Maxwell field and so Einstein's field equations are given by

$$R_{ab} = \alpha \ell_a \ell_b \tag{1}$$

where α is a global smooth real-valued function and ℓ a global smooth nowhere zero null vector field on M. The null vector field ℓ represents the rays of the null Maxwell field and Maxwell's equations then show that ℓ is geodesic and shear-free. In calculations ℓ will be taken as locally scaled so that it is affinely parametrized.

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2. Conformal symmetries

Let X be a global conformal vector field on M. Then one has

$$X_{a;b} = \phi g_{ab} + F_{ab} \qquad (\Leftrightarrow \pounds_X g_{ab} = 2\phi g_{ab}) \tag{2}$$

where ϕ is the *conformal scalar* of X, F is the *conformal bivector* of X, a semi-colon denotes a covariant derivative and $\pounds$ a Lie derivative. In keeping with the usual geometrical interpretation of a conformal vector field, X should be C^3 and then it follows that X (and ϕ and F) are necessarily smooth. Equation (2) also implies that F and ϕ satisfy

$$F_{ab;c} = R_{abcd}X^d - 2\phi_{[a}g_{b]c} \tag{3}$$

and

$$\phi_{a;b} = -\tfrac{1}{2}R_{ab;c}X^c - \phi R_{ab} + R_{c(a}F_{b)}{}^c \tag{4}$$

where round and square brackets denote the usual symmetrization and skew-symmetrization, respectively, where ϕ_a denotes the gradient of ϕ and where the vanishing of the Ricci scalar (from (1)) has been incorporated into (4). Throughout, X is assumed to be 'properly' conformal in the sense that it is not homothetic over any non-empty open subset of M. In other words, *ϕ_a cannot vanish over any non-empty open subset of M.*

If the Weyl tensor vanishes everywhere on M then it follows that M is locally isometric to a conformally flat 'generalized' (Einstein–Maxwell) plane wave (see e.g. Kramer *et al* 1980) and the (local) conformal vector fields arising can be completely constructed (see e.g. Hall *et al* 1992). The question of their global existence then depends on global topological requirements (Hall 1989) and so this case is essentially known. Although the Weyl tensor will, in general, vanish at some points of M and not at others, it will be assumed here, in keeping with the local nature of the present paper, that *the Weyl tensor vanishes at no point of M*. The Weyl tensor is necessarily of an algebraically special Petrov type with ℓ spanning a repeated principal null direction at each point of M (Goldberg and Sachs 1962) and the finite number of such directions ($\leqslant 2$ since C_{abcd} is never zero on M) together with equation (2) and the consequent result $\pounds_X C^a{}_{bcd} = 0$ shows that

$$\pounds_X \ell^a = \mu \ell^a \qquad (\Leftrightarrow X_{a;b}\ell^b - \ell_{a;b}X^b = \mu \ell_a) \tag{5}$$

for some function μ on M. The bracketed equation in (5) together with (2) gives

$$\phi\ell_a + F_{ab}\ell^b - \mu\ell_a = \ell_{a;b}X^b \tag{6}$$

and then use of (6) and (1) in (4) gives

$$\phi_{a;b} = \nu\ell_a\ell_b \tag{7}$$

for some function ν on M. The Riemann tensor is now introduced through the Ricci identity for the covector field ϕ_a which, when contracted with ℓ and use is made of (7), gives

$$R_{abcd}\ell^c\phi^d = 0. \tag{8}$$

Finally, using the usual expression for the Weyl tensor in terms of the Riemann and Ricci tensors together with equations (1) and (8) one finds

$$C_{abcd}H^{cd} = 0 \tag{9}$$

where H is the bivector given by $H_{ab} = 2\ell_{[a}\phi_{b]}$.

Let U be the (necessarily open) subset of M on which ϕ_a is not proportional to ℓ_a and let $V = M\backslash U$. Now V contains the zeros of ϕ_a and because of the original assumption regarding these zeros the open subset $\tilde{V}$, defined as the intersection of the interior of V, int V, and the open dense subset of M where ϕ_a is not zero, is such that $V\backslash\tilde{V}$ has no interior. It now follows that the closed subset $M\backslash(U \cup \tilde{V})$ of M has empty interior. Now, on U the bivector H is nowhere zero and (9) shows that the Weyl tensor has a zero eigenvalue everywhere on U and is hence of Petrov type N or III everywhere on U. On $\tilde{V}$ one has ϕ_a nowhere zero and proportional to ℓ_a, and (7) shows that ℓ must be recurrent (i.e. $\ell_{a;b} = \ell_a q_b$ for some covector field q_a) on this subset. It then follows from an elementary argument using the Ricci identity and (1) that $\ell^a C_{abcd} = \ell_b P_{cd}$ for some bivector P which satisfies $\ell_{[b}P_{cd]} = 0$ (and hence is simple with ℓ in its blade) and $P_{ab}\ell^b = 0$ (and so is a null bivector with principal null direction spanned by ℓ). It now follows (Bel 1962) that the Weyl tensor is of Petrov type N or III on $\tilde{V}$. Thus the Petrov type is N or III over the *open dense* subset $U \cup \tilde{V}$ of M and hence everywhere on M (since if it were either of the other two admissible types II or D at some at some $p \in M$ it would be so over some open neighbourhood of p). So let $W \subseteq M$ be the (necessarily open) subset of M where the Petrov type is III. It should be noted that since the only eigenbivectors of the Weyl tensor for this type are null, H is necessarily a null bivector and hence $\phi_a\ell^a = 0$ on W.

Suppose W is not empty and let $p \in W$. There exists an open neighbourhood A of p in W and a smooth null tetrad (ℓ, n, x, y) on A where ℓ is the null vector field described earlier and where $\ell^a n_a = x^a x_a = y^a y_a = 1$ with all other tetrad inner products zero. Since ℓ is affinely parametrized the shearfree condition on ℓ gives

$$\ell_{a;b}x^a x^b = \ell_{a;b}y^a y^b \qquad \ell_{a;b}x^a y^b = -\ell_{a;b}y^a x^b. \tag{10}$$

Now the Petrov type III condition allows a canonical expression for C_{abcd} to be written down in A after, possibly, adjusting the choice of n, x and y in the above null tetrad (which will not affect (10)), in the form

$$C_{abcd} = b(N_{ab}M_{cd} + M_{ab}N_{cd} - \overset{*}{N}_{ab}\overset{*}{M}_{cd} - \overset{*}{M}_{ab}\overset{*}{N}_{cd}) \tag{11}$$

where b is a nowhere zero function on A, $N_{ab} = 2\ell_{[a}x_{b]}$, $M_{ab} = 2\ell_{[a}n_{b]}$ and * denotes the usual duality operator. Equations (1) and (11) then lead to a similar canonical form for the curvature tensor on A

$$R_{abcd} = a(N_{ab}N_{cd} + \overset{*}{N}_{ab}\overset{*}{N}_{cd}) + b(N_{ab}M_{cd} - \overset{*}{N}_{ab}\overset{*}{M}_{cd} + M_{ab}N_{cd} - \overset{*}{M}_{ab}\overset{*}{N}_{cd}) \tag{12}$$

where a is some function on A. This latter canonical form together with equation (7) can then be directly substituted into the Ricci identity for ϕ_a and the resulting equation contracted successively with $x^a x^b$, $y^a y^b$, $x^a y^b$ and $y^a x^b$ (and noting that $\phi_a\ell^a = 0$ on W). The expressions obtained, when combined with (10) (and considering separately the cases $\nu(p) = 0$, $\nu(p) \neq 0$) show that at each $p \in A$, $\phi_a x^a = \phi_a y^a = 0$. However $\phi_a\ell^a = 0$ on W and so $\phi_a \propto \ell_a$. Recalling the clause about the vanishing of ϕ_a it now follows from (7) that ℓ is recurrent on A. Hence *ℓ is recurrent on the whole of W*. It is *properly* recurrent there (i.e. it cannot be rescaled so as to be (locally) covariantly constant). Otherwise, the Ricci identity would give $R_{abcd}\ell^d = 0$ and hence $C_{abcd}\ell^d = 0$ in the relevant region and thus the Petrov type would be N. Such metrics generalize those of Goldberg and Kerr mentioned earlier and, as mentioned in the introduction, cannot be vacuum metrics if a proper conformal vector field is admitted. Further discussion of such spacetimes together with a specific example of one which admits a proper conformal vector field is given in appendix 1.

Now let W' be the open subset of M given by $W' = \mathrm{int}(M \backslash W)$ so that the Petrov type is N everywhere on W'. Then one repeats the calculation of the previous paragraph but where now the appropriate canonical forms for the Weyl and curvature tensors in some open neighbourhood of any $p \in M$ are

$$C_{abcd} = c(N_{ab}N_{cd} - \overset{*}{N}_{ab}\overset{*}{N}_{cd}) \qquad R_{abcd} = dN_{ab}N_{cd} + e\,\overset{*}{N}_{ab}\overset{*}{N}_{cd} \tag{13}$$

where c is a nowhere zero function and d and e functions on that neighbourhood. One obtains

$$\nu(\ell_{a;b}x^a x^b) = \nu(\ell_{a;b}y^a y^b) = \nu(\ell_{a;b}x^a y^b) = \nu(\ell_{a;b}y^a x^b) = 0. \tag{14}$$

If D is the (necessarily open) subset of W on which ν is non-zero, equation (14) shows that ℓ *has vanishing twist expansion and shear on* D. On the interior (in W') of the set $W' \backslash D$, $\nu \equiv 0$ and so, from (7), $\phi_{a;b} = 0$ on $\mathrm{int}(W' \backslash D)$. If ϕ_a is not null at one (and hence every) point of $\mathrm{int}(W' \backslash D)$ one obtains an immediate contradiction with the algebraic type of the Ricci tensor (1) (Hall 1990). Thus ϕ_a is null everywhere on $\mathrm{int}(W' \backslash D)$. The Ricci identity applied to ϕ_a then gives $R_{abcd}\phi^d = 0$ (and hence, from (1), $\phi_a \ell^a = 0$). It follows that $\phi_a \propto \ell_a$ on $\mathrm{int}(W' \backslash D)$ and hence from (7) that ℓ_a is recurrent on this set. Returning to the subset D, if the rotation τ of ℓ (in the sense of Ehlers and Kundt 1962) fails to vanish at any point p and hence in some open neighbourhood E of p then g would restrict to one of Kundt's type N class of metrics on E (Kundt 1961). But then it follows (see appendix 2 and cf Salazar *et al* (1983) for the vacuum case) that the conformal vector field X is *homothetic* when restricted to E, contradicting the proper conformal nature of X. It follows that the rotation of ℓ is zero on D and hence (since its expansion twist and shear also vanish on D) that ℓ is recurrent on D (cf Ehlers and Kundt 1962). Using the previous argument one now sees that ℓ is recurrent on W'. Thus $\ell_{a;b} = \ell_a q_b$ and the type N condition on the Weyl tensor together with (1) yields $R_{abcd}\ell^d = 0$ and hence, from the Ricci identity, $q_{[a;b]} = 0$. It follows that q is locally the gradient of a function σ, say, and that ℓ may be locally rescaled to the vector field $\mathrm{e}^{-\sigma}\ell$ which is covariantly constant; $(\mathrm{e}^{-\sigma}\ell_a)_{;b} = 0$. Thus (W', g) is locally isometric to a generalized (null Einstein–Maxwell) pp-wave.

In summary, the following result has been established.

Theorem. Let (M, g) be a null Einstein–Maxwell field whose Weyl tensor does not vanish at any point of M and which admits a proper conformal vector field (equation (2) with ϕ_a not vanishing over any non-empty open subset of M). Then M may be decomposed disjointly as $M = W \cup (M \backslash W)$ where W is open and the Petrov type is III at each point of W and where the Petrov type is N at each point of $M \backslash W$. The null ray vector ℓ is properly recurrent on W and scalable to a covariantly constant null vector field on $\mathrm{int}(M \backslash W)$. The region $\mathrm{int}(M \backslash W)$ is everywhere locally isometric to a generalized pp-wave.

Appendix 1

Here it will be shown that on the open subset W of M where the Weyl tensor is of Petrov type III everywhere and ℓ is recurrent, the function ϕ is necessarily a function of u (u being the coordinate adapted such that $\ell_a = -u_{,a}$). An example of a null fluid with these characteristics (Petrov type III, ℓ recurrent) which admits a proper conformal vector field will also be provided. It follows from the above requirements that the metric belongs to one of Kundt's class of metrics (Kundt 1961) and it can be written in the form

$$ds^2 = 2\,d\zeta\,d\bar{\zeta} - 2\,du(dv + w\,d\zeta + \bar{w}\,d\bar{\zeta} + H\,du) \tag{A1}$$

in some open neighbourhood of any point of W. Here the notation is that of Kramer *et al* (1980, p 295) with $\ell^a = (0,1,0,0)$, $\ell_a = (-1,0,0,0)$ and

$$w = w(\bar{\zeta}, u) \qquad H = \tfrac{1}{2}(w_{,\bar{\zeta}} + \bar{w}_{,\zeta})v + H^0. \tag{A2}$$

It is remarked that for the other family of solutions listed in Kramer *et al*, ℓ is non-recurrent.

If the metric in (A1) admits a proper conformal vector field X with associated function ϕ as in (2) then (5) gives

$$X^{\zeta}_{,v} = X^{\bar{\zeta}}_{,v} = X^{u}_{,v} = 0 \tag{A3}$$

and (2) can be rewritten as

$$H_{,c}X^c + 2HX^u_{,u} + X^v_{,u} + wX^{\zeta}_{,u} + \bar{w}X^{\bar{\zeta}}_{,u} = 2\phi H \tag{A4}$$

$$X^u_{,u} + X^v_{,v} = 2\phi \tag{A5}$$

$$X^u_{,\zeta} = X^u_{,\bar{\zeta}} = 0 \tag{A6}$$

$$X^{\zeta}_{,\bar{\zeta}} = X^{\bar{\zeta}}_{,\zeta} = 0 \tag{A7}$$

$$X^{\zeta}_{,\zeta} + X^{\bar{\zeta}}_{,\bar{\zeta}} = 2\phi \tag{A8}$$

$$w_{,c}X^c + X^v_{,\zeta} + w(X^u_{,u} + X^{\zeta}_{,\zeta}) - X^{\bar{\zeta}}_{,u} = 2\phi w \tag{A9}$$

$$\bar{w}_{,c}X^c + X^v_{,\bar{\zeta}} + \bar{w}(X^u_{,u} + X^{\bar{\zeta}}_{,\bar{\zeta}}) - X^{\zeta}_{,u} = 2\phi\bar{w}. \tag{A10}$$

From (A3) and (A6) it readily follows that $X^u = X^u(u)$; and from (A3) and (A8) that $\phi_{,v} = 0$, and thus (A5) gives

$$X^v = (2\phi - X^u_{,u})v + C(u, \zeta, \bar{\zeta}) \tag{A11}$$

where C is a real function of its arguments. Substituting (A11) into (A9) and (A10), and taking into account (A7) it follows that $\phi_{,\zeta} = \phi_{,\bar{\zeta}} = 0$; i.e. $\phi = \phi(u)$ and also

$$X^{\zeta} = A(u)\zeta + B(u) \qquad C = \alpha(u)(\zeta + \bar{\zeta}) + \beta(u). \tag{A12}$$

Differentiating (A9) with respect to $\bar{\zeta}$ and (A10) with respect to ζ, and adding gives

$$\bar{w}_{,u\zeta} + w_{,u\bar{\zeta}} + w_{,\bar{\zeta}\bar{\zeta}}(A\bar{\zeta} + B) + \bar{w}_{,\zeta\zeta}(A\zeta + B) - 2A_{,u} = 0 \tag{A13}$$

which constitutes an integrability condition for (A9) and (A10). Substituting now (A13), (A15) and (A16) into (A4), one finds from the coefficients of the terms linear in v,

$$(\bar{w}_{,u\zeta} + w_{,u\bar{\zeta}})X^u + w_{,\bar{\zeta}\bar{\zeta}}(A\bar{\zeta} + B) + \bar{w}_{,\zeta\zeta}(A\zeta + B) + (\bar{w}_{,\zeta} + w_{,\bar{\zeta}})X^u_{,u} = -4A_{,u} + 2X^u_{,uu}. \tag{A14}$$

Combining the two last equations gives

$$A_{,u} = \tfrac{1}{3}X^u_{,uu} \qquad \text{i.e. } A = \tfrac{1}{3}X^u_{,u} + \text{constant}. \tag{A15}$$

Solutions to the above equations can be found, thus providing examples of Petrov type III null fluid spacetimes admitting proper conformal vector fields; take, for instance, w and H^0 in (A1) to be

$$w = u^{-4}\bar{\zeta}^2 + u^{-1} + u^{-2} \qquad \text{and} \qquad H^0 = -w\bar{w} \tag{A16}$$

then this metric admits the following conformal vector field X

$$X = u^3\partial_u - u^2 v\partial_v + u^2\zeta\partial_{\zeta} + u^2\bar{\zeta}\partial_{\bar{\zeta}} \tag{A17}$$

with corresponding conformal factor $\phi = u^2$.

Appendix 2

Suppose now that M is everywhere of Petrov type N, and that the rotation of ℓ fails to vanish at some point $p \in M$ and hence in some open neighbourhood E of p (i.e. ℓ is non-recurrent on E). It will now be shown that X is homothetic on E.

Choosing local coordinates on E the line element can be written as in (A1) with the functions w and H being in this case (see Kramer *et al*, p 295):

$$w = -2v/(\zeta + \bar{\zeta}) \qquad \text{and} \qquad H = -v^2/(\zeta + \bar{\zeta})^2 + H^0(u, \zeta, \bar{\zeta}). \tag{A18}$$

Substituting these values in the equations (A3)–(A10), it follows from (A9), (A5) and (A8) that

$$\phi = \phi(u) \qquad X^{\zeta} = \phi(u)\zeta \tag{A19}$$

$$X^{v} = (2\phi - X^{u}_{,u})v + C(u, \zeta, \bar{\zeta}) \tag{A20}$$

where C is a real function satisfying

$$-2C/(\zeta + \bar{\zeta}) + C_{,\zeta} - \phi_{,u}\bar{\zeta} = 0. \tag{A21}$$

Substituting (A19) and (A20) back into (A4), one finds from the linear terms in v that

$$C = -\tfrac{1}{2}X^{u}_{,uu}(\zeta + \bar{\zeta})^2 \tag{A22}$$

and hence (A21) implies

$$\phi_{,u} = 0 \tag{A23}$$

and so X is homothetic on E.

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