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Some developments on axial symmetry

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Abstract. The definition of axial symmetry in general relativity is reviewed, and some results concerning the geometry in a neighbourhood of the axis are derived. Expressions for the metric are given in different coordinate systems, and emphasis is placed on how the metric coefficients tend to zero when approaching the axis.

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1. Introduction

The purpose of this paper is to review, clarify and prove some new results concerning axial symmetry in general relativity.

Apart from its intrinsic mathematical interest, axial symmetry is physically significant in general relativity in that it is believed to describe quite accurately a large number of situations of interest in the astrophysical context, or even (up to a certain degree of approximation) in cosmology, see for instance [1].

In what is to follow, (M, g) will denote a spacetime; i.e. M is a connected, C^3 Hausdorff four-dimensional manifold and g is a C^2-Lorentz metric of signature +2.

The paper is structured as follows. In section 2 we begin by pointing out some of the unsatisfactory characteristics of the usual 'polar' coordinates r and ϕ, give the definition of axial symmetry and summarize its most immediate consequences. In so doing, we shall partly follow the work of Mars and Senovilla [2, 3], which brilliantly generalizes, reviews and updates the pioneering work on axial symmetry by Carter [4, 5] and Martín [6]; we shall provide alternative proofs to the statements in [2, 3], mostly based on the fixed-point theorems for conformal Killing vectors due to Hall [10] (see [11] as well), and also prove the existence of certain geometric structures which will be of help for later developments. We shall relate these results to those by Wilson and Clarke [7], where a theory for a particular class of axially symmetric spacetimes which are regular on the axis is put forward, and the condition of 'elementary flatness' on the axis is thoroughly studied in connection with other conditions such as the trivial limiting holonomy of a family of loops and the regularity of various degrees of differentiability. Note, though, that in this reference a certain form for the metric is assumed, implying strong restrictions on the axial Killing vector field (hypersurface orthogonality and spacelike character everywhere except on the axis, among others); whereas the present treatment, based on the definition of axial symmetry as given in [2, 3], covers a much broader class of spacetimes (see the next section). Section 3 looks deeper into the geometric

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structure of axially symmetric spacetimes in a neighbourhood of the axis. Coordinates with a well defined geometrical meaning are introduced, and the behaviour of the metric coefficients near the axis is worked out in this and in other related coordinate systems; giving coordinate expressions for the metric in those systems which could be of interest for the study of this class of spacetimes.

In section 4, we briefly discuss the case of axially symmetric spacetimes admitting another isometry, and consider the special cases of stationary and cylindrically symmetric spacetimes. Finally, in section 5, we present a 'fluid toy model' possessing axial symmetry.

2. Axial symmetry revisited

The intuitive idea of axial symmetry (see, e.g., [1], p 192) is that of an isometry generated by a spacelike Killing vector (KV), say $\vec{\xi}$, whose orbits are closed (compact) curves. The *axis of symmetry* is then the set of points which are unchanged by the isometry, or *fixed points*, which can be shown to be precisely those at which $\vec{\xi}$ vanishes.

The usual approach then consists in adapting a coordinate, say ϕ, to the KV generating the axial isometry so that $\vec{\xi} = \partial_\phi$. This has the advantage that the metric and other geometrical objects in the manifold become independent of this coordinate, but strictly speaking, one should be aware that a coordinate chart with one of the coordinates chosen in this way, can never contain points belonging to the symmetry axis, and therefore this choice may not be the most convenient one when it comes to studying geometrical or physical properties in a neighbourhood of the axis. Furthermore, this coordinate choice can be misleading in some other ways; consider for instance $\mathbb{R}^2$ endowed with the usual Euclidean flat metric whose associated line element reads, in the usual Cartesian coordinates (x, y):

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2;$$

the axial Killing vector is $\vec{\xi} = y\partial_x - x\partial_y$ and the axis of symmetry (fixed points) consists of a single point O (the origin) with coordinates $x = y = 0$; note that $\vec{\xi}(\mathrm{O}) = 0$, whereas $\xi_{a;b}(\mathrm{O}) \neq 0$ as it should be since otherwise a well known theorem (see, for example, [1], p 100) would imply that $\vec{\xi} = 0$ everywhere on $\mathbb{R}^2$. Consider next the polar coordinates (ρ, ϕ) defined as usual; the line element then reads

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2,$$

$\vec{\xi}$ becomes $\vec{\xi} = \partial_\phi$, and the fixed point O is $\rho = 0$. Now computing $\xi_a(\mathrm{O})$ and $\xi_{a;b}(\mathrm{O})$ in these coordinates, it turns out they are both zero, which would seem to imply (see above) that $\vec{\xi} = 0$ everywhere in $\mathbb{R}^2$, which is obviously not true. Needless to say, this is just a coordinate problem, because the Jacobian of the change of coordinates we have performed is singular at O (note also that the metric is apparently singular at $\rho = 0$); or, in other words, the coordinate chart (ρ, ϕ) does not cover O†. These considerations apply to the case of any space or spacetime with axial symmetry such that a coordinate has been adapted to the axial KV in the above form and contains points belonging to the symmetry axis.

The above problems (coordinate singularities and the need to distinguish them from real ones, and 'bad' behaviour of the axial Killing vector on the axis) call for a precise definition of axial symmetry and a detailed study of its consequences, which in turn implies stepping

† If one starts by considering a two-dimensional manifold with metric $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2$, the point $\rho = 0$ does *not* belong to the manifold, hence, strictly speaking, the above considerations do not apply, what happens in this case is that there exists a trivial extension of the manifold which also includes $\rho = 0$. I thank Professor Senovilla for clarifying this point.

initially away from cylindrical coordinates to recover them later on in the process of writing down the metric.

The remainder of this section will be devoted to presenting such a definition and providing an account of its most immediate consequences. We shall follow the definition of axial symmetry as given by Mars and Senovilla [2] (see also [3]), and we shall prove the results derived from that definition by making use of some powerful theorems on fixed points of conformal vector fields due to Hall (see [10] for an excellent account of these results). We also refer the reader to [2, 3] for alternative proofs to most of our statements.

Definition 1. *A spacetime (M, g) is said to have axial symmetry if and only if there is an effective realization of the one-dimensional torus T into M that is an isometry and such that its set of fixed points is non-empty.*

One remark is in order here: definition 1 implicitly assumes that there exists at least one fixed point in (M, g). If this condition is dropped, the spacetime is said to be *cyclically symmetric* (see [5]). Examples of cyclically symmetric spacetimes do indeed exist, and can be constructed by identifying points in spacetimes admitting a spacelike isometry (see [8]). Alternatively, one could consider, for example, the exterior field of an axially symmetric source whenever the axis is entirely contained in the source, or also the case of Misner spacetime [9]. However, as Carter showed [5], cyclically symmetric spacetimes which are asymptotically Minkowskian in spacelike directions, have necessarily fixed points under the isometry and are therefore axially symmetric.

Also note that definition 1 does not coincide with the definition of axial symmetry given by Carter in the above reference [5], as the set of fixed points (or axis) was required to be a two-dimensional surface there, but as is stated explicitly in that reference, the existence of fixed points in cyclically symmetric spacetimes readily implies that they form a two-dimensional surface (see proposition 3 in [5]), hence both definitions are, in fact, equivalent once the distinction between cyclic and axial symmetry is made.

In what follows, the one-parameter group of axial isometries (effective realization of T) will be denoted as $\{\varphi_t, t \in T\}$ and the KV $\vec{\xi}$ that generates it will be referred to as the *axial KV*, whereas its (non-empty) set of zeros (fixed points of the isometry) will be denoted by W_2 and called *axis of symmetry*, i.e.

$$W_2 \equiv \{p \in M : \vec{\xi}(p) = 0\}. \tag{1}$$

Coordinate indices will be denoted by lowercase latin letters and the covariant derivative by a semicolon; Killing's equation will then read

$$\xi_{a;b} + \xi_{b;a} = 0.$$

The *Killing bivector*, F_{ab}, is defined as $F_{ab} \equiv \xi_{a;b}$. Note that at points $p \in W_2$, $F_{ab}(p) \neq 0$ necessarily (otherwise $\vec{\xi} = 0$, see remark above), and since a KV is a particular case of affine motion it satisfies $\xi_{a;bc} = R_{abcd}\xi^d$; that is, $F_{ab;c}(p) = 0$. Summarizing,

$$\forall p \in W_2, \qquad \xi^a(p) = F_{ab;c}(p) = 0, \qquad F_{ab}(p) \neq 0. \tag{2}$$

The above considerations, together with the fact that the axial Killing vector has closed periodic orbits, have powerful implications on the geometry of the spacetime in a neighbourhood of the axis, as we shall see presently. To this end, let p be a point on the axis ($p \in W_2$) and consider the tangent space at p, T_pM; the map φ_{t*} is then an automorphism of T_pM, given by [10, 12] $\varphi_{t*} = \exp(tA)$, where A is the matrix $A = \xi^a{}_{,b}(p)$ computed in any coordinate system. Note that since $\xi^a(p) = 0$, $\xi^a{}_{,b}(p) = F^a{}_b(p)$; that is, matrix A coincides with the automorphism associated with the Killing bivector at p. Let ψ now denote the exponential diffeomorphism from some open neighbourhood of $\vec{0} \in T_pM$ to some open neighbourhood U of $p \in W_2$; it follows that [12]

$$\psi \circ \varphi_{t*} = \varphi_t \circ \psi, \tag{3}$$

wherever it makes sense.

Bearing all of this in mind, we can prove the following

Theorem 1. *Let p be a fixed point of the axial isometry, then the Killing bivector is spacelike at p.*

Proof. The proof to this statement can be largely gathered from the results on fixed points of conformal symmetries due to Hall [10]. Thus, a Killing bivector at a fixed point can be non-simple (eigenvalues $\pm\beta, \pm i\alpha$, type E), timelike (eigenvalues $\pm\beta, 0, 0$, type E), null (all eigenvalues equal to 0, type B2), or spacelike (eigenvalues $0, 0, \pm i\alpha$, type C2). Since we demand the isometry to have closed, periodic orbits, it follows that none of the integral curves of $\vec{\xi}$ can become arbitrarily close to the fixed point p, which rules out type E in Hall's classification and therefore $F_{ab}(p)$ cannot be non-simple, nor timelike. Assume now that $F_{ab}(p)$ is null, it then follows that $F_{ab}(p) = 2l_{[a}x_{b]}$ for some null vector $\vec{l}$ and some unit spacelike vector $\vec{x}$ orthogonal to $\vec{l}$; the differential map at p would be, in a null tetrad containing $\vec{l}$ and $\vec{x}$:

$$\varphi_{t*}|_p = \exp(tA); \quad \Rightarrow \quad (\varphi_{t*}|_p)^a{}_b = \delta^a{}_b + t(l^a x_b - x^a l_b) + \tfrac{1}{2}t^2 l^a l_b,$$

and one readily sees that there is no value of t, other than $t = 0$, for which $\varphi_{t*}|_p = id|_{T_pM}$; from where it follows that $F_{ab}(p)$ must be spacelike. □

Since $F_{ab}(p)$ is spacelike, one has $F_{ab}(p) = \lambda(x_a y_b - y_a x_b)$ for some spacelike, unit, mutually orthogonal vectors $\vec{x}$ and $\vec{y}$. The differential map at p will be, in a null tetrad $\{\vec{l}, \vec{n}, \vec{x}, \vec{y}\}$ $(l^a n_a = x^a x_a = y^a y_a = 1)$,

$$\begin{aligned}(\varphi_{t*}|_p)^a{}_b &= \delta^a{}_b + tF^a{}_b(p) + \tfrac{1}{2}t^2 F^a{}_c F^c{}_b(p) + \cdots \\ &= \delta^a{}_b + \left[-\frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} - \cdots\right](x^a x_b + y^a y_b) \\ &\quad + \left[\lambda t - \frac{(\lambda t)^3}{3!} + \cdots\right](x^a y_b - y^a x_b) \\ &= l^a n_b + n^a l_b + \cos(\lambda t)(x^a x_b + y^a y_b) + \sin(\lambda t)(x^a y_b - y^a x_b);\end{aligned} \tag{4}$$

and one can now, without loss of generality, rescale t so as to obtain the standard 2π periodicity, thus having $\lambda = 1$ and hence

$$F_{ab}(p) = x_a y_b - y_a x_b. \tag{5}$$

Now let $P_p \subset T_pM$ be the subspace spanned by $\{\vec{x}, \vec{y}\}$; its image under the exponential diffeomorphism ψ will be a two-dimensional, regular, spacelike submanifold of U, N_p, which on account of (3) is mapped onto itself by the axial isometry. Thus, the orbits of $\vec{\xi}$ are 'packed' around the axis forming the submanifolds N_p, the axial KV $\vec{\xi}$ being tangent to them; hence

Proposition 1. *The axial Killing vector is spacelike on U.*

Similarly, the image by ψ of the subspace $L_p \subset T_pM$ spanned by $\{\vec{l}, \vec{n}\}$ (i.e. the vectors spanning the blade of the dual bivector of $F_{ab}(p)$) is also a two-dimensional, regular, timelike submanifold of U whose points are fixed under the action of the isometry which can be seen to be totally geodesic (see [12]) and, since the connection is symmetric, autoparallel.

Thus, we have proven

Theorem 2. *The axis of symmetry, W_2, is a two-dimensional, timelike autoparallel surface.*

Since $\psi_* = id|_{T_pM}$, it follows that the subspaces L_p and P_p introduced above are precisely the subspace tangent to the axis at p and its orthogonal complement, respectively; i.e. $L_p = T_pW_2$, and $P_p = (T_pW_2)^{\perp}$, and from equation (4) it can be immediately seen that (see also [2]):

Theorem 3. *For any point p on the axis, if $\vec{v}_p \in T_pM$ ($\vec{v}_p \neq 0$), then*

(a) $\vec{v}_p \in L_p \Leftrightarrow \varphi_{t*}(\vec{v}_p) = \vec{v}_p, \forall t$. *(Alternatively, $\vec{v}_p$ is tangent to the axis iff $[\vec{v}, \vec{\xi}]|_p = 0$).*

(b) $\vec{v}_p \in P_p \Leftrightarrow \varphi_{t*}(\vec{v}_p) = \vec{w}_p, \forall t$, *such that $\vec{v}_p$ and $\vec{w}_p$ are linearly independent but $\varphi_{t*}(\vec{w})$ depends linearly on the previous two. (Alternatively, $\vec{v}|_p$ is normal to the axis at p iff $\vec{v}|_p$ and $[\vec{v}, \vec{\xi}]|_p$ are linearly independent and $[[\vec{v}, \vec{\xi}], \vec{\xi}]|_p$ depends linearly on the previous one.)*

(c) $\vec{v}_p \notin L_p, \vec{v}_p \notin P_p \Leftrightarrow \varphi_{t*}(\vec{v}_p) = \vec{w}_p, \forall t$, *is such that $\vec{v}_p$, $\vec{w}_p$ and $\varphi_{t*}(\vec{w}_p) = \vec{u}_p$ are linearly independent vectors, but $\varphi_{t*}(\vec{u}_p)$ depends linearly on the previous one. (Alternatively, $\vec{v}$ is neither tangent nor normal to the axis at p iff $\vec{v}_p$, $[\vec{v}, \vec{\xi}]|_p$ and $[[\vec{v}, \vec{\xi}], \vec{\xi}]|_p$ are linearly independent vectors and $[[[\vec{v}, \vec{\xi}], \vec{\xi}], \vec{\xi}]|_p$ depends linearly on the previous one.)*

Further results concern the Petrov and Segre types of Weyl and Ricci tensors at points on the axis (see [10]). They can be easily obtained by taking into account that both the Ricci and the Weyl tensors are invariant under isometries. Considering then $\mathcal{L}_{\vec{\xi}} R_{ab} = 0$ and $\mathcal{L}_{\vec{\xi}} C_{abcd} = 0$ at points p on the axis, and since $\vec{\xi}(p) = 0$, these Lie derivatives read simply $R_{cb}F^c{}_a + R_{ac}F^c{}_b \stackrel{p}{=} 0$ (i.e. $F_{ab}(p)$ determines an eigenspace of the Ricci tensor at p), and $C_{ebcd}F^e{}_a + C_{aecd}F^e{}_b + C_{abed}F^e{}_c + C_{abce}F^e{}_d \stackrel{p}{=} 0$, from which the algebraic types can be readily read off, thus giving:

Theorem 4. *In an axially symmetric spacetime and at points on the axis, the Petrov type of the Weyl tensor can only be D or O, whereas the Segre type of the Ricci tensor is* $\{2, (11)\}$, $\{1, 1(11)\}$, $\{z\bar{z}, (11)\}$† *or some degeneracy thereof.*

A further consequence of definition 1 above is the so-called *elementary flatness condition*, which ensures the standard 2π-periodicity of the axial coordinate near the axis [2, 3] (see also [7] for an alternative proof and an interesting study of its relation to other properties).

Theorem 5. *At points near the axis of symmetry one has.*

$$\left.\frac{\nabla_c(\xi_a\xi^a)\nabla^c(\xi_a\xi^a)}{4\xi_a\xi^a}\right|_{W_2} \longrightarrow 1. \tag{6}$$

† As is well known, the Segre type $\{z\bar{z}, 11\}$ cannot satisfy the dominant energy condition and therefore is not of physical interest.

3. Geometry in a neighbourhood of the axis. Coordinate considerations

In this section we will study the behaviour of the metric when the symmetry axis W_2 (assumed to be regular) is approached. We will do this by introducing various coordinate systems on the exponential neighbourhood defined in the preceding section, and discussing their geometrical meaning.

Now let $p \in W_2$ be a fixed point of the isometry, from (3) it can be immediatly seen that in the resulting exponential coordinate system in U, say $\{x^a\}$, the axial KV $\vec{\xi}$ has components

$$\xi^a = F^a{}_b(p)x^b; \tag{7}$$

that is, they are linear functions of the coordinates† $\{x^a\}$. Consider now the regular submanifold $N_p \subset U$ through p and choose (normal) coordinates x, y on it such that $x(p) = y(p) = 0$. It follows that there exist coordinates (x, y, z, t) on U such that, for any $p' \in W_2 \cap U$, points in $N_{p'} = \psi_{p'}(P_{p'})$ have all the same z and t coordinates and $x(p') = y(p') = 0$; furthermore, in this coordinate system we have‡:

Proposition 2. *In the coordinates introduced on U, the axial Killing vector reads*

$$\vec{\xi} = y\partial_x - x\partial_y. \tag{8}$$

The existence of one such coordinate system can be seen as follows: for any $q' \in U$, $q' \notin W_2, q' \notin N_p$, there exists a point $p' \in W_2 \cap U$ such that $q' \in N_{p'} \cap U$. Now let γ be the geodesic (entirely contained) in $W_2 \cap U$ joining p and p', and let τ denote the parallel transport along γ from p to p'. Next choose a null tetrad at p, $\{\vec{l}_p, \vec{n}_p, \vec{x}_p, \vec{y}_p\}$ such that $\{\vec{l}_p, \vec{n}_p\}$ and $\{\vec{x}_p, \vec{y}_p\}$ span L_p and P_p, respectively. We can define a tetrad field on γ by parallel transporting the above tetrad along it. Since W_2 is autoparallel, it follows that $\{\vec{l}, \vec{n}\}$ and $\{\vec{x}, \vec{y}\}$ will span at each point on $W_2 \cap U$ the tangent space to the axis at that point and its orthogonal complement, respectively (see [12], vol II, p 60). Finally, choose normal coordinates x, y on $N_p \cap U$ as above (which, in particular, can be such that $\partial_x|_p \parallel \vec{x}_p$ and $\partial_y|_p \parallel \vec{y}_p$) and define $\chi : N_p \cap U \to N_{p'} \cap U$ as $q' = \chi(q) \equiv (\psi_{p'} \circ \tau \circ \psi_p^{-1})(q)$; the map χ then defines coordinates on $N_{p'}$ with the required properties.

Note that equation (8) is actually a requirement in the definition of C^k-regularity as given in [7], the other condition being that, in those coordinates, the metric and its inverse have C^k components on some neighbourhood of the axis, from which the elementary flatness condition follows if $k \geqslant 2$ (see proposition 1 in [7]). Our point of view is slightly different in that assuming the metric to be C^2 in some neighbourhood of the axis we actually prove the existence of coordinates in which $\vec{\xi}$ takes the required form. Furthermore, and despite the fact that these coordinates are non-unique as we shall discuss later on in this paper, the coordinates x, y introduced have a clear geometric significance; namely, they are normal coordinates on the regular submanifolds which are perpendicular to the axis at every point of it, whose existence we have also proven.

In what follows, we shall put $x^A = \{x, y\}$, $A = 1, 2$ and $x^\alpha = \{z, t\}$, $\alpha = 3, 4$; hence the submanifolds N are simply those given by $x^\alpha =$ constant, whereas the axis W_2 is given by $x^A = 0$. It can then be shown

† Equations (3) and (7) as well as (2) and the discussion following, hold for any affine motion, including homotheties and proper affine motions.
‡ For some of the developments that follow, it might be necessary to restrict U to a *convex normal neighbourhood* $U' \subset U$; however, we shall keep writing U so as not to complicate the notation unnecessarily.

Theorem 6. *In the previously established notation it follows that*

$$g_{xx} \stackrel{W_2}{=} g_{yy}, \qquad g_{xy} \stackrel{W_2}{=} 0, \qquad g_{A\alpha} \stackrel{W_2}{=} 0, \tag{9}$$

$$g_{xx,A} \stackrel{W_2}{=} g_{yy,A} \stackrel{W_2}{=} 0, \qquad g_{xy,d} \stackrel{W_2}{=} 0, \qquad g_{xx,\alpha} \stackrel{W_2}{=} g_{yy,\alpha}, \tag{10}$$

$$g_{\alpha\beta,A} \stackrel{W_2}{=} 0, \qquad g_{\alpha x,x} \stackrel{W_2}{=} g_{\alpha y,y}, \qquad g_{\alpha x,y} \stackrel{W_2}{=} -g_{\alpha y,x}. \tag{11}$$

Proof. All of the above results can be seen to follow from $F^a{}_{b;c} \stackrel{W_2}{=} 0$ (see equation (2)), the form of the axial KV on U given by (8); i.e. $\vec{\xi} = y\partial_x - x\partial_y$, from where one also has

$$F^a{}_b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

on U, and the equations

$$\mathcal{L}_{\vec{\xi}} g_{ab} = 0, \qquad \text{and} \qquad \partial_d(\mathcal{L}_{\vec{\xi}} g_{ab}) = 0, \tag{13}$$

both evaluated on the axis W_2.

Alternatively, some of the above can be derived from

$$\Gamma^A_{\alpha\beta} \stackrel{W_2}{=} \Gamma^\alpha_{\beta A} \stackrel{W_2}{=} 0, \tag{14}$$

which is just a direct consequence of the autoparallel character of W_2†. Also note that

$$\Gamma^A_{BC} \stackrel{W_2}{=} \Gamma^\alpha_{AB} \stackrel{W_2}{=} 0, \tag{15}$$

which shows the 'normal' character of the coordinates x^A chosen on N. □

The significance of the preceding theorem lies in that it yields information on how certain metric coefficients tend to zero when approaching the symmetry axis, thus helping to understand the meaning of this and other related coordinate systems that one can use in a neighbourhood of the axis.

Let us next write down the most general form for the metric on U in the above coordinates $x^a = (x, y, z, t)$. Imposing $\mathcal{L}_{\vec{\xi}} g = 0$ with $\vec{\xi}$ given by (8) it follows:

$$g_{ab} = \begin{pmatrix} B + A\sin 2(\phi + M) & A\cos 2(\phi + M) & D\sin(\phi + N) & E\sin(\phi + S) \\ A\cos 2(\phi + M) & B - A\sin 2(\phi + M) & D\cos(\phi + N) & E\cos(\phi + S) \\ D\sin(\phi + N) & D\cos(\phi + N) & F & J \\ E\sin(\phi + S) & E\cos(\phi + S) & J & H \end{pmatrix}, \tag{16}$$

where $\phi \equiv \arctan x/y$ and $A, B, D, E, F, J, M, N, S$ and H are all functions of t, z and $r \equiv (x^2 + y^2)^{1/2}$. The function r is invariant on each orbit of $\vec{\xi}$, thus labelling the orbits of the axial KV on each submanifold N_p.

† See [12] vol II, p 53 for the definition of an autoparallel surface from where the first part of equation (14) follows directly, and [12], vol II, p 60 for a general property of autoparallel surfaces stating that every vector orthogonal to an autoparallel surface remains orthogonal to it under parallel transport along any curve contained in that surface, which is equivalent to the second part above. This can be seen to be equivalent to the vanishing of the two second fundamental forms on the axis W_2.

It is quite interesting to see what theorem 6 implies for the various metric functions and the way they tend (or do not tend) to zero when approaching the symmetry axis W_2: note that, since $g_{A\alpha} \overset{W_2}{=} 0$ (see (9), (11)) implies that E and D tend to zero when r does as, at least, r; furthermore, and without loss of generality, if t and z are chosen orthogonal on W_2, then (11) implies that J must tend to zero as r^2 at least. On the other hand, equations (9) and (10) imply that A also tends to zero as r^2 at least, whereas B must be of the form $B = B_0(x^\alpha) + \mathrm{O}(r^2)$ with $B_0(x^\alpha) \neq 0$. As for the other metric coefficients, regularity on W_2 together with (11) implies that the functions F and H must be such that $F = F_0(x^\alpha) + \mathrm{O}(r)$ and $H = H_0(x^\alpha) + \mathrm{O}(r)$, respectively, with $F_0(x^\alpha) \neq 0$, $H_0(x^\alpha) \neq 0$.

So far we have not made use of the coordinate freedom we still have. Note that a rotation in the x, y plane such as

$$x' = r\sin(\phi + h(r, z, t)), \tag{17}$$

$$y' = r\cos(\phi + h(r, z, t)); \tag{18}$$

which preserves the form of the axial KV $\vec{\xi}$ (i.e. $\vec{\xi} = y'\partial_{x'} - x'\partial_{y'}$), the submanifolds N, the axis (i.e. W_2 is given by $x' = y' = 0$), and the form of the metric (including the behaviour near the axis of E, D and J), allows us to set $A = 0$, i.e. $g_{xy} = 0$; alternatively, the above transformation can be used to set M, N or S equal to zero.

The above transformation is a rotation on each of the regular submanifolds N which also depends on r, that is, on the particular orbit of $\vec{\xi}$.

If we now change to polar coordinates r, ϕ defined as above, the KV takes then the familiar form

$$\vec{\xi} = \frac{\partial}{\partial\phi}, \tag{19}$$

while the metric reads, in coordinates $x^{a'} = (\phi, r, z, t)$ as

$$g_{ab} = \begin{pmatrix} r^2(B + A\sin 2M) & rA\cos 2M & rD\sin N & rE\sin S \\ rA\cos 2M & B - A\sin 2M & D\cos N & E\cos S \\ rD\sin N & D\cos N & F & J \\ rE\sin S & E\cos S & J & H \end{pmatrix}, \tag{20}$$

where again, a redefinition of the angular coordinate such as $\phi \to \phi + h(r, z, t)$ would allow us to set $A = 0$ (or one of the functions M, N or S), but we choose not do so at this stage. Note the extra factors r and r^2 in some of the metric coefficients. Thus, taking into account our previous comments on how the different functions tend to zero when approaching the axis, we have that (at least)

$$\begin{aligned} &g_{\phi\phi} \sim \mathrm{O}(r^2), \quad && g_{\phi r} \sim \mathrm{O}(r^3), \quad && g_{\phi z} \sim \mathrm{O}(r^2), && \\ &g_{\phi t} \sim \mathrm{O}(r^2), && g_{rr} \sim \mathrm{O}(0), && g_{rz} \sim \mathrm{O}(r), && \\ &g_{rt} \sim \mathrm{O}(r), && g_{zz} \sim \mathrm{O}(0), && g_{zt} \sim \mathrm{O}(r^2), \quad && g_{tt} \sim \mathrm{O}(0). \end{aligned} \tag{21}$$

We shall next perform coordinate changes so as to bring the above metric to other forms, better suited for some kinds of calculations (such as, for instance, numerical calculations). In doing so, two things will be of major interest to us. On one hand, we want the time coordinate to remain completely free (i.e. we do not want to make use of the gauge freedom in choosing the coordinate time at this stage; again, this is of interest in numerical calculations based on the 1 + 3-formalism of general relativity). On the other hand, we want to 'keep track' of how the metric coefficients tend to zero when approaching the axis (this is crucial when computing derivatives numerically near the axis in spacelike directions). To this end, let us consider the effect of the following family of coordinate transformations; namely,

$$\begin{aligned} x' &= f(r, x^\beta)\sin(\phi + h(r, x^\beta)), & y' &= f(r, x^\beta)\cos(\phi + h(r, x^\beta)), \\ z' &= G(r, x^\beta), & t' &= t, \end{aligned} \tag{22}$$

or, equivalently in terms of polar coordinates:

$$\phi' = \phi + h(r, x^\beta), \qquad r' = f(r, x^\beta), \qquad z' = G(r, x^\beta), \qquad t' = t.$$

These changes preserve the form of the axial KV (i.e. $\vec{\xi} = y'\partial_{x'} - x'\partial_{y'}$), but they do not preserve the coordinate expression of the submanifolds N (i.e. $N_p = \{x^\alpha = \text{constant}\} \neq \{x^{\alpha'} = \text{constant}\}$). The symmetry axis W_2 is preserved if and only if $f(r, x^\beta)$ is such that $f(0, x^\beta) = 0$. The form of the metric is also preserved, and the behaviour near the axis of the metric coefficients changes depending on how the functions $f(r, x^\beta)$ and $G(r, x^\beta)$ are chosen. Since we are interested in preserving the coordinate expression of the axis, let us put

$$f(r, x^\beta) = r\bar{f}(r, x^\beta) \qquad \text{with} \quad \bar{f}(0, x^\beta) \neq 0; \tag{23}$$

and also, without loss of generality

$$G(z, x^\beta) = z + \tilde{G}(r, x^\beta). \tag{24}$$

Note that the inverse change of coordinates, will also be of the above form, that is: $\phi = \phi' + \hat{h}(r', x^{\beta\prime}), r = \hat{f}(r', x^{\beta\prime}), z = G(r', x^{\beta\prime}), t = t'$, and also $\hat{f} = r'\bar{\hat{f}}(r', x^{\beta\prime})$ with $\bar{\hat{f}}(0, x^{\beta\prime}) \neq 0$ and $z = z' + \tilde{\hat{G}}(r', x^{\beta\prime})$.

One can now work out the expressions of the metric coefficients in the new (primed) coordinates, along with their behaviour near the axis. The calculation is straightforward but rather long and tedious, we give the results dropping primes for convenience:

$$\begin{aligned} &g_{\phi\phi} \sim \mathrm{O}(r^2), && g_{\phi r} \sim \min \mathrm{O}(r^3, r^2 h_{,r}, r^2\tilde{G}_{,r}), \\ &g_{\phi z} \sim \mathrm{O}(r^2), && g_{\phi t} \sim \mathrm{O}(r^2), \qquad g_{rr} \sim \mathrm{O}(0), \\ &g_{rz} \sim \mathrm{O}(r), && g_{rt} \sim \mathrm{O}(r), \qquad g_{zz} \sim \mathrm{O}(0), \\ &g_{zt} \sim \min(\mathrm{O}(r^2), \mathrm{O}(G_{,t})), && g_{tt} \sim \mathrm{O}(0). \end{aligned} \tag{25}$$

The following comments are now in order.

(a) If $h_{,r} \stackrel{W_2}{=} G_{,r} \stackrel{W_2}{=} 0$, then $g_{\phi r} \sim \mathrm{O}(r^3)$.
(b) If one sets $G(r, z, t) = z + r^2\bar{G}(r, z, t)$ with $\bar{G}(0, z, t) \neq 0$ (i.e. $\tilde{G} = r^2\bar{G}$), then $g_{zt} \sim \mathrm{O}(r^2)$.
(c) Note that, by implementing one such coordinate change, we can, in fact, extend the neighbourhood U of the axis to which all our previous discussions were restricted; thus the new (primed) coordinate chart can be defined on $V \supseteq U$, x' and y' no longer being normal coordinates on N.

It is now easy to show that if we perform one of the above changes with

$$\phi' = \phi + h(r, x^\beta), \qquad r' = r\bar{f}(r, x^\beta), \qquad z' = z + r^2\bar{G}(r, x^\beta), \qquad t' = t; \tag{26}$$

where $\bar{f}(0, x^\beta) \neq 0$ and $\bar{G}(0, x^\beta) \neq 0$†, the metric takes the form (dropping primes)

$$g_{ab} = \begin{pmatrix} r^2\bar{g}_{\phi\phi} & r^2\bar{g}_{\phi r} & r^2\bar{g}_{\phi z} & r^2\bar{g}_{\phi t} \\ r^2\bar{g}_{\phi r} & \bar{g}_{rr} & r\bar{g}_{rz} & r\bar{g}_{rt} \\ r^2\bar{g}_{\phi z} & r\bar{g}_{rz} & \bar{g}_{zz} & r^2\bar{g}_{zt} \\ r^2\bar{g}_{\phi t} & r\bar{g}_{rt} & r^2\bar{g}_{zt} & \bar{g}_{tt} \end{pmatrix}, \tag{27}$$

where $\bar{g}_{ab} = \bar{g}_{ab}(r, z, t)$, and they are non-zero on the axis $r = 0$‡. The above form of the metric is invariant under the coordinate changes given by (26), and so are the expressions of both the axial KV ($\xi = \partial_\phi$), and the axis W_2 ($r = 0$).

Further, the functions h, $\bar{f}$ and $\bar{G}$ in (26) can be chosen so that (dropping primes again) $g_{\phi t} = g_{rt} = g_{zt} = 0$, thus the metric reads

$$g_{ab} = \begin{pmatrix} r^2\bar{g}_{\phi\phi} & r^2\bar{g}_{\phi r} & r^2\bar{g}_{\phi z} & 0 \\ r^2\bar{g}_{\phi r} & \bar{g}_{rr} & r\bar{g}_{rz} & 0 \\ r^2\bar{g}_{\phi z} & r\bar{g}_{rz} & \bar{g}_{zz} & 0 \\ 0 & 0 & 0 & \bar{g}_{tt} \end{pmatrix}, \tag{28}$$

where, as before, the barred functions are non-zero on the axis $r = 0$ (see the footnote referred to above).

We shall refer to the above form of the metric as the *shift-free form* because it corresponds to $g^{ti} = 0$, $i = \phi, r, z$, that is, the so-called shift vector of the metric in the $1 + 3$-formalism is zero.

The easiest way to prove this, is by showing that it is always possible, by means of one of the above transformations, to set the shift vector equal to zero, i.e.

$$g^{t'\phi'} = g^{t\phi} + g^{tt}h_{,t} + g^{tr}h_{,r} + g^{tz}h_{,z} = 0, \tag{29}$$

$$g^{t'r'} = g^{tt}r\bar{f}_{,t} + g^{tr}(\bar{f} + r\bar{f}_{,r}) + g^{tz}r\bar{f}_{,z} = 0, \tag{30}$$

$$g^{t'z'} = g^{tt}r^2\bar{G}_{,t} + g^{tr}(2r\bar{G} + r^2\bar{G}_{,r}) + g^{tz}(1 + r^2\bar{G}_{,z}) = 0. \tag{31}$$

Now, from the fact that g^{ti}, $i = \phi, r, z$ depend on r, z and t and the elementary theory of differential equations, it can be immediately seen that one can always find a function $h(r, z, t)$ satisfying (29). On the other hand, equations (30) and (31) can be rewritten as

$$r(g^{tt}\gamma_{,t} + g^{tr}\gamma_{,r} + g^{tz}\gamma_{,z}) = -g^{tr}, \tag{32}$$

$$2rg^{tr}\bar{G} + r^2(g^{tt}\bar{G}_{,t} + g^{tr}\bar{G}_{,r} + g^{tz}\bar{G}_{,z}) = -g^{tz}, \tag{33}$$

where $\gamma \equiv \ln \bar{f}$, and again the theory of differential equations, together with the expressions of g^{tt}, g^{tr} and g^{tz} in terms of the metric functions appearing in (20) and their respective orders as given in (21), allow us to conclude that solutions to the above equations always exist for some coordinate ranges (see the appendix for details).

† If one imposes that on the axis $\frac{\partial x^{A'}}{\partial x^B} = \delta^A_B$, it follows that $\bar{f}(r, z, t) \overset{W_2}{=} 1$ and also $h(r, z, t) \overset{W_2}{=} 0$. Furthermore, if the 'normal' character (see (15)) of the coordinates on N is to be preserved; i.e. $\Gamma^{A'}_{B'C'} \overset{W_2}{=} \Gamma^{\alpha'}_{B'C'} \overset{W_2}{=} 0$, it also follows that $\bar{f}_{,r}(r, z, t) \overset{W_2}{=} h_{,r}(r, z, t) \overset{W_2}{=} 0$, and then $g_{\phi r} = \mathrm{O}(r^3)$, see (25).

‡ This is necessarily so for the diagonal terms $\bar{g}_{aa}$ in order for the metric to be regular at points on the axis; as for the non-diagonal terms $\bar{g}_{ab}$ with $a \neq b$, they could also vanish on the axis in particular spacetimes. The above is just a 'lower bound' on how quickly the metric coefficients tend to zero when approaching the axis.

It is interesting to note that this will only be possible, in general, if the spacetime admits no further isometries which form a group together with the axial isometry. Let the spacetime be (for instance) stationary as well as axially symmetric; one can then always adapt the time coordinate t to the Killing vector implementing the stationarity (see the next section for details) and the metric reads then as in (20), the metric functions depending now only on r and z. In this case though, it is not possible in general to perform one of the above changes of coordinates that simultaneously preserves the form of the timelike Killing vector and sets $g_{t'\phi'}$, $g_{t'r'}$ and $g_{t'z'}$ equal to zero†. To see this, consider the above equations (29)–(31) specialized to this case, they read (without now even restricting the form of f and G)

$$g^{t'\phi'} = g^{t\phi} + g^{tr}h_{,r} + g^{tz}h_{,z} = 0, \tag{34}$$

$$g^{t'r'} = g^{tr}f_{,r} + g^{tz}f_{,z} = 0, \tag{35}$$

$$g^{t'z'} = g^{tr}G_{,r} + g^{tz}G_{,z} = 0. \tag{36}$$

Since all the functions involved depend now only on r and z, it follows that in order for the last two equations to be satisfied, the gradients of f and G should be linearly dependent at each point, the coordinate change then being non-admissible. The same holds if the spacetime admits any other isometry which, together with the axial isometry, forms a group G_2 (see the next section for further details).

4. Axially symmetric spacetimes admitting further symmetries

In this section, and for the sake of completeness, we summarize some results on spacetimes admitting other isometries which, together with the axial one, form a two-parameter group of isometries G_2. Essentially, all the results we present in this section are known, and the reader is referred to [2, 3, 13, 14] for the proofs omitted here, as well as for more detailed discussions.

The basic result concerning this issue, was already known to relativists some three decades ago but, surprisingly, it has been forgotten and rediscovered many times over [15]; it can be stated as follows.

Theorem 7. *Let (M, g) be an axially symmetric spacetime‡ (axial KV $\vec{\xi}$) admitting another Killing vector field $\vec{\lambda}$ such that $\vec{\xi}, \vec{\lambda}$ generate a two-parameter group of isometries G_2, it then follows that*

$$[\vec{\xi}, \vec{\lambda}] = 0, \tag{37}$$

that is, G_2 is Abelian.

Proof. Let $\vec{\xi}, \vec{\lambda}$ be the infinitesimal generators of the isometries φ_t and χ_s, respectively, and let φ_t be such that $\varphi_0(x) = \varphi_{2\pi}(x) = x$ (or equivalently, $\varphi_t(x) = \varphi_{t+2\pi}(x)$ for any $x \in M$); i.e. the orbits of the subgroup $G_1 \equiv \{\varphi_t\}$ spanned by $\vec{\xi}$ are closed.

Let O_1 be the (closed) orbit of G_1 through a given point $x_0 \in M$, that is $\mathrm{O}_1 = \{\varphi_t(x_0), \forall t \in [0, 2\pi)\}$, and let χ_s be an infinitesimal isometry generated by $\vec{\lambda}$. Consider now $\chi_s(\mathrm{O}_1)$, which will also be closed since χ_s is a diffeomorphism. In any coordinate chart x^a covering O_1 and $\chi_s(\mathrm{O}_1)$, we will have $\chi_s(\mathrm{O}_1) = \{x^a(t) + s\lambda^a(x^b(t)) + \mathrm{O}(s^2), x^a(t) = \varphi_t^a(x_0)\}$; the fact that

† This will only be possible when the spacetime is static, that is, the timelike Killing vector is hypersurface orthogonal.
‡ This result holds for any two-parameter group of transformations of a spacetime such that the orbits of one of their subgroups are closed.

$\chi_s(\mathrm{O}_1)$ is closed implies that for any value of the parameter t, $x^a(t+2\pi)+s\lambda^a(x^b(t+2\pi)) = x^a(t)+s\lambda^a(x^b(t))$, and since $x^a(t+2\pi) = x^a(t)$. This, in turn, implies

$$\lambda^a(x^b(t+2\pi)) = \lambda^a(x^b(t)) \tag{38}$$

for points on O_1.

Now let $x \in \mathrm{O}_1$ such that $x^a = x^a(t)$ and let $\varphi^a_{2\pi}(x) = y^a$. Next define on O_1 the vector field $\vec{\lambda}'$ by Lie-dragging $\vec{\lambda}$ along O_1, i.e.

$$\vec{\lambda}'(x) \equiv \exp(-t\mathcal{L}_{\vec{\xi}})\,\vec{\lambda}(x). \tag{39}$$

Since $\lambda'^c(y) = \frac{\partial y^c}{\partial x^a}\lambda^a(x)$, it follows from (38) that $\vec{\lambda}'(\varphi_{2\pi}(x)) = \vec{\lambda}(\varphi_{2\pi}(x))$ for any $x \in \mathrm{O}_1$, which, on account of (39) can be expressed as

$$\exp(-2\pi\mathcal{L}_{\vec{\xi}})\,\vec{\lambda}(x) = \vec{\lambda}(x), \tag{40}$$

expanding the first member above we have

$$\vec{\lambda}(x) - 2\pi[\vec{\xi},\vec{\lambda}](x) + \frac{(2\pi)^2}{2!}[\vec{\xi},[\vec{\xi},\vec{\lambda}]](x)\cdots = \vec{\lambda}(x). \tag{41}$$

Now, if $\vec{\xi},\vec{\lambda}$ generate a group, it must be that $[\vec{\xi},\vec{\lambda}] = a\,\vec{\xi} + b\,\vec{\lambda}$ for some constants a and b, and substituting this into the above equation one has

$$-2\pi(a\vec{\xi}+b\vec{\lambda}) + \frac{(2\pi)^2}{2!}b(a\vec{\xi}+b\vec{\lambda})\cdots = 0, \tag{42}$$

which readily implies $a = b = 0$ from where the theorem follows. □

From the preceding theorem and theorem 3, it follows that any such Killing vector field $\vec{\lambda}$ must be tangent to the axis W_2.

Suppose now for definiteness that the orbits of the Abelian G_2 generated by $\vec{\xi},\vec{\lambda}$ are timelike (T_2 in the following) over a certain region of $W_2 \cap U$; it then follows that a timelike Killing vector must exist which can be set equal to $\vec{\lambda}$ without loss of generality, and it can then be immediately seen that one can choose the time coordinate adapted to it, i.e. $\vec{\lambda} = \partial_t$. The other coordinate on $W_2 \cap U$, z, can still be introduced in a way such that $g(\partial_t,\partial_z)|_{W_2} = 0$, and the form of the metric in the coordinates $\{x, y, z, t\}$ chosen on U, but with t adapted to $\vec{\lambda}$, will be that of (16) with all the arbitrary functions appearing there now depending on $r = (x^2+y^2)^{1/2}$ and z alone; note though that the time coordinate t is no longer free. Similar comments apply to coordinates $\{\phi, r, z, t\}$ and the form of the metric (20).

Coordinate changes such as (26), but with the functions h, $\bar{f}$ and $\bar{G}$ not depending on t, will render the metric (20) (independent of t) in the form (see comments following (25))

$$g_{ab} = \begin{pmatrix} r^2\bar{g}_{\phi\phi} & r^2\bar{g}_{\phi r} & r^2\bar{g}_{\phi z} & r^3\bar{g}_{\phi t} \\ r^2\bar{g}_{\phi r} & \bar{g}_{rr} & r\bar{g}_{rz} & r^2\bar{g}_{rt} \\ r^2\bar{g}_{\phi z} & r\bar{g}_{rz} & \bar{g}_{zz} & r^2\bar{g}_{zt} \\ r^3\bar{g}_{\phi t} & r^2\bar{g}_{rt} & r^2\bar{g}_{zt} & \bar{g}_{tt} \end{pmatrix}, \tag{43}$$

with $\bar{g}_{ab} = \bar{g}_{ab}(r,z)$, $\bar{g}_{ab}(0,z) \neq 0$, and simultaneously preserve the form of both Killing vector fields ($\vec{\xi} = \partial_\phi$, $\vec{\lambda} = \partial_t$) and the coordinate expression of the axis ($r = 0$). Again, the above form of the metric, the expressions of both Killing fields and the axis of symmetry, are invariant under the above coordinate changes; thus, choosing $h(r,z)$, $\bar{f}(r,z)$ and $\bar{G}(r,z)$ appropriately the metric (43) can be brought to the Weyl form [1], but in general not to the shift-free form (28) unless $\vec{\lambda}$ happens to be hypersurface orthogonal (see the comments at the end of the preceding section).

The case in which the orbits of G_2 are spacelike (S_2), corresponds to cylindrical symmetry; a spacetime is said to be *cylindrically symmetric* if and only if it admits a group G_2 of isometries acting on spacelike orbits S_2 and contains an axial isometry. The reader is referred to [13, 14] for a detailed discussion, but comments regarding the form of the metric, etc similar to those in the T_2 case apply also here.

5. A fluid 'toy model'

The purpose of this section is to build a simple model of axially symmetric perfect fluid using the shift-free form of the metric (28) introduced in section 3 and imposing the following restrictions.

(a) The barred functions in (28) do not depend on r; that is, the radial dependence is factored out in the form of the coefficients appearing in that expression. This, in particular, will allow one to set $g_{tt} = -1$ without loss of generality.
(b) The spacetime presents a discrete symmetry $z \mapsto -z$ (reflection across the equatorial plane); that is, $g_{\phi z} = g_{rz} = g_{tz} = 0$.

From the above restrictions and (28) it follows that the form of the metric is

$$g_{ab} = \begin{pmatrix} r^2\bar{g}_{\phi\phi}(z,t) & r^2\bar{g}_{\phi r}(z,t) & 0 & 0 \\ r^2\bar{g}_{\phi r}(z,t) & \bar{g}_{rr}(z,t) & 0 & 0 \\ 0 & 0 & \bar{g}_{zz}(z,t) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{44}$$

Demanding the Einstein tensor associated with the above metric to be of Segre type [1] {(111), 1} and excluding further degeneracies (Λ-term), we shall note its non-degenerate timelike (unit) eigenvector (4-velocity of the fluid) as $\vec{u}$, ($g(\vec{u},\vec{u}) = -1$), its associated eigenvalue as $-\rho$, $\rho \geqslant 0$ (where ρ is the energy density as measured by an observer comoving with the fluid), and the degenerate eigenvalue by p (the pressure measured by that same observer); Einstein's field equations will then read

$$G^a_b = (\rho + p)u^a u_b + p\delta^a_b. \tag{45}$$

Since $\vec{u}$ is non-degenerate, it follows that it must be invariant under the isometry group; that is,

$$[\vec{\xi}, \vec{u}] = 0; \tag{46}$$

hence, theorem 3 implies that $\vec{u}$ must be tangent to the axis of symmetry W_2 at points on it. This implies, using coordinates $\{x, y, z, t\}$ defined as in (16) that $u^x \overset{W_2}{=} u^y \overset{W_2}{=} 0$, or equivalently $u^r \overset{W_2}{=} u^\phi \overset{W_2}{=} 0$; and from (45) it then follows that

$$G^\phi_r \overset{W_2}{=} G^\phi_z \overset{W_2}{=} G^\phi_t \overset{W_2}{=} G^r_\phi \overset{W_2}{=} G^r_z \overset{W_2}{=} G^r_t \overset{W_2}{=} 0. \tag{47}$$

Further, from the discrete symmetry across the equatorial plane $z = 0$, it also follows that $u^z = 0$, and the field equations (45) then imply

$$G^z_\phi = G^z_r = G^z_t = 0. \tag{48}$$

A direct computation of G^z_ϕ and G^z_r for the metric (44) implies, on account of (48) that

$$\bar{g}_{\phi r} = \alpha(t)\bar{g}_{\phi\phi} \qquad \bar{g}_{rr} = \beta(t)\bar{g}_{\phi\phi}. \tag{49}$$

Substituting this into the expressions of G^ϕ_t and G^r_t, and taking into account (47), one obtains

$$\alpha_{,t} = \beta_{,t} = 0; \tag{50}$$

that is, α and β are both constant, whereas $G^\phi_r \overset{W_2}{=} G^r_\phi \overset{W_2}{=} 0$ are satisfied identically.

Now setting $\bar{g}_{\phi\phi}(z,t) \equiv A(z,t)$, $\bar{g}_{\phi r}(z,t) \equiv B(z,t)$, $\bar{g}_{rr}(z,t) \equiv P(z,t)$ and $\bar{g}_{zz}(z,t) \equiv M(z,t)$, it follows from $G^z_t = 0$ (see (48)) that

$$\frac{A_{,tz}}{A} - \frac{1}{2}\frac{A_{,z}}{A}\frac{A_{,t}}{A} - \frac{1}{2}\frac{A_{,z}}{A}\frac{M_{,t}}{M} = 0, \tag{51}$$

from where it follows that

$$M = \frac{(A_{,z})^2}{A}, \tag{52}$$

where an arbitrary function of z has been set equal to one by suitably redefining the coordinate z.

At this stage it can be immediately seen that the fluid 4-velocity is just $\vec{u} = \partial_t$ (comoving velocity), and $G^\phi_\phi = G^r_r$ is satisfied identically, whereas $G^\phi_\phi = G^z_z$ holds if and only if $\alpha = 0$, which casts the metric into diagonal form and allows one to set $\beta = 1$ by a trivial redefinition of the radial coordinate. The line element then reads

$$\mathrm{d}s^2 = A(z,t)\left\{r^2\,\mathrm{d}\phi^2 + \mathrm{d}r^2 + \left(\frac{A_{,z}}{A}\right)^2 \mathrm{d}z^2\right\} - \mathrm{d}t^2, \tag{53}$$

where the function $A(z,t)$ must satisfy

$$\frac{1}{2}\frac{A_{,t}}{A}\left(\frac{A_{,tz}}{A_{z}} - \frac{A_{,t}}{A}\right) + \frac{A_{,tt}}{A} - \frac{A_{,ttz}}{A_{z}} - \frac{1}{4A} = 0. \tag{54}$$

The energy density ρ and pressure p are given by

$$\rho = -\frac{1}{4A} - \frac{1}{4}\left(\frac{A_{,t}}{A}\right)^2 + \frac{A_{,t}}{A}\frac{A_{,tz}}{A_{z}}, \qquad p = \frac{1}{4A} + \frac{1}{4}\left(\frac{A_{,t}}{A}\right)^2 - \frac{A_{,tt}}{A}. \tag{55}$$

Note that the above metric (53) admits three independent spacelike Killing vectors (the axial KV, $\vec{\xi} = \partial_\phi$, $\vec{\eta} = \sin\phi\,\partial_r + (1/r)\cos\phi\,\partial_\phi$, and $\vec{\zeta} = \cos\phi\,\partial_r - (1/r)\sin\phi\,\partial_\phi$) acting on two-dimensional flat orbits (surfaces of constant t and z), and is therefore a type B warped spacetime (see [16]).

6. Conclusions

In this paper, we have reviewed the definition of axial symmetry for a spacetime and derived some of its most immediate consequences assuming that the axis is non-singular. Many of these results were already known (see [2–6]), but we have provided alternative proofs based on fixed-point theorems for conformal Killing vectors (see [10]); some others, such as the existence of submanifolds N consisting of isometry orbits 'packed' together in a neighbourhood of the axis and its consequences (see the next paragraph), do not seem to appear in the literature on the subject.

We next introduced a coordinate system with a well defined geometrical meaning and gave the expressions of both the axial KV and the metric in those coordinates; further, and taking into account geometrical properties of the submanifolds N and the symmetry axis W_2, we worked out how the metric coefficients tend to zero (for those which do) when the axis is approached. We then discussed the coordinate changes that preserve the form of both the metric and the axial KV, as well as the location of the symmetry axis, giving again, in these new coordinates, the way in which the metric coefficients tend to zero when approaching the axis. Using one such coordinate change, we have shown that it is always possible to transform the metric to the *shift-free form* given by equation (28), provided that no other isometry exists in the spacetime which forms a group G_2 with the axial isometry.

Attention was then devoted to the consequences of the existence of another Killing vector which, along with the axial KV, spans a group G_2. Special mention has been made of the cases of stationary and cylindrically symmetric spacetimes.

In the last section, a 'toy model' of a fluid possessing axial symmetry was presented.

Appendix

Let us consider the form (20) of the metric with $A = 0$ without loss of generality. One then has

$$(\det g)g^{tt} = FB^2 - r^2\bar{D}^2[\sin(2M + 2N) - B], \tag{A1}$$

$$(\det g)g^{tr} = -r\bar{E}FB\cos S + r^3\bar{D}\bar{J}B\cos N + r^3\bar{D}^2\bar{E}\sin N\sin(N - S), \tag{A2}$$

$$(\det g)g^{tz} = -r^2\bar{J}B^2 + r^2\bar{D}\bar{E}B\cos(N - S); \tag{A3}$$

where we have put $D = r\bar{D}$, $E = r\bar{E}$ and $J = r^2\bar{J}$, the barred quantities being functions of r, z and t which do not vanish (in general, that is, except in particular spacetimes) on the axis $r = 0$. It can then be immediately seen that equations (32) and (33) always have a solution of the required characteristics.

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