Conformal symmetries in warped manifolds

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Abstract.   The existence of a Conformal Vector Field (CVF) is studied in the important class of warped manifolds of arbitrary dimension generalizing in this way the corresponding results of the four dimensional geometries. As a concrete example we apply the geometric results in the case of brane-world scenarios when the bulk geometry admits a hypersurface orthogonal Killing Vector Field (KVF) and is filled with a perfect fluid matter content.

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1. Introduction

Warped Lorentzian manifolds $(\mathcal{M}, \mathbf{g})$ are characterized by the existence of two submanifolds $\mathcal{O}, \mathcal{S}$ with associated metrics $\mathbf{g}_1, \mathbf{g}_2$ and appropriate signature in order to maintain the overall Lorentzian signature of $\mathbf{g}$ and a function Y defined on $\mathcal{O}$ such that $\mathcal{M} = \mathcal{O} \times \mathcal{S}$ and $\mathbf{g} = \mathbf{g}_1 \otimes Y^2 \mathbf{g}_2$. In the case where $n = 4$ the resulting construction leads to the notion of a warped product spacetime (the reader is referred to [1, 2] for a discussion on the geometric and dynamical properties of warped product spacetimes). Both in classical relativity and in higher dimensional theories of gravitation, important solutions of the Field Equations (FE) are of warped form providing a strong motivation to study the implications either by the existence of a specific symmetry assumption or by assuming a specific matter source for the gravitational field or even the possible interplay between the two assumptions.

The propose of the present article is to develop a methodology of studying the existence of Conformal Vector Fields (CVFs) $\mathbf{X}$ in the important class of warped geometries irrespective of the dimension and the specific geometric structure of the constituent submanifolds. We recall that CVFs are defined by the requirement $\mathcal{L}_{\mathbf{X}} \mathbf{g} = 2\psi \mathbf{g}$ where ψ is a smooth function of the warped manifold and reduce to Killing Vector Fields (KVFs) when $\psi = 0$, to Homothetic Vector Fields (HVFs) for $\psi_{;A} = 0$ and to Special CVFs for $\psi_{;AB} = 0$. Throughout this work the following index conventions have been used: full $(d + m)$−dimensional indices are denoted by capital latin letters $A, B, ... = 0, 1, 2, ..., d + m$ $(d, m \neq 1)$, lower latin letters from the first half of the latin alphabet denote spacetime indices $a, b, ... = 0, 1, 2, 3$, lower latin letters from the second half correspond to m−dimensional coordinates $i, j, ... = d + 1, d + 2, ..., m$ and greek indices take the values $\alpha, \beta, ... = 1, 2, ..., d$.

2. Conformal symmetries of $d + m$ decomposable metrics

The main result of the present work is the next Theorem which generalizes known results [3, 4, 5] in the literature concerning the existence of CVFs in 4-dimensional decomposable space-times,

in the case of a $(d+m)-$decomposable manifold.

Theorem 1. *Let $g_{AB} = g_{\alpha\beta} \otimes g_{ij}$ be a $(d+m)-$decomposable metric with Lorentzian signature. Then:*

(i) *The KVFs of each metric $g_{\alpha\beta}$, g_{ij} are also KVFs of the full $(d+m)-$metric g_{AB}.*

(ii) *The metric g_{AB} admits a HVF iff each $g_{\alpha\beta}$, g_{ij} admits one. If $H_\alpha(x^\beta)$ and $\bar{H}_i(x^j)$ are HVFs of the constituent metrics $g_{\alpha\beta}$, g_{ij} with conformal factors b and $\bar{b}$ respectively, the HVF H_A of the $(d+m)-$metric is given by $H_A = \bar{b}H_\alpha\delta_A^\alpha + b\bar{H}_i\delta_A^i$ with conformal factor $b\bar{b}$.*

(iii) *A necessary condition for the $(d+m)-$metric to admit a proper CVF with conformal factor ψ is each of the constituent metrics to admit the gradient CVFs $\psi_{,\alpha}, \psi_{,i}$.* $\square$

Proof

It is convenient to write the $d+m$ decomposable metric as follows:

$$g_{AB} = g_{\alpha\beta}(x^\gamma)\delta_A^\alpha\delta_B^\beta + g_{ij}(x^k)\delta_A^i\delta_B^j \tag{2.1}$$

and we denote the covariant derivative with respect to the metrics $g_{\alpha\beta}$, g_{ij} with "|" and "||" respectively.

For general vectors $K_\alpha(x^A), K_i(x^A)$ we decompose their first derivatives into irreducible parts, according to the general relations:

$$K_{\alpha|\beta} = \lambda(x^A)g_{\alpha\beta} + H_{\alpha\beta}(x^A) + F_{\alpha\beta}(x^A) \tag{2.2}$$

$$K_{i||j} = \bar{\lambda}(x^A)g_{ij} + \bar{H}_{ij}(x^A) + \bar{F}_{ij}(x^A) \tag{2.3}$$

where $H_{\alpha\beta} = H_{\beta\alpha}$, $H_\alpha^\alpha = 0$, $F_{\alpha\beta} = -F_{\beta\alpha}$ and similarly for the "i,j" indices.

Let us consider now a general vector field ξ_A in the $d+m$ manifold $(\mathcal{M}, \mathbf{g})$. Denoting the covariant derivative with respect to g_{AB} by a semicolon we write again:

$$\xi_{A;B} = \psi(x^C)g_{AB} + H_{AB}(x^C) + F_{AB}(x^C). \tag{2.4}$$

For convenience we $d+m$ decompose the general vector field ξ_A as follows:

$$\xi_A = K_\alpha\delta_A^\alpha + \bar{K}_i\delta_A^i \tag{2.5}$$

where $K_\alpha, \bar{K}_i$ are the orthogonal projections of ξ_A.

From equations (2.2) -(2.5) and following [3] we may express the irreducible parts of ξ_A w.r.t. to the corresponding parts of the vector fields $K_\alpha, \bar{K}_i$. We restrict our considerations to ξ^A being a CVF of the metric g_{AB} which is equivalent to $H_{AB} = 0$. It follows that:

$$H_{\alpha\beta} = 0, \qquad \bar{H}_{ij} = 0, \qquad \lambda = \bar{\lambda} = \psi, \qquad K_{\alpha,j} + \bar{K}_{j,\alpha} = 0 \tag{2.6}$$

Because $\Gamma_{\alpha i}^A = 0$, (2.6) can be written:

$$K_{\alpha||j} + \bar{K}_{j|\alpha} = 0. \tag{2.7}$$

Using the fact that K_α is a CVF of the metric $g_{\alpha\beta}$ differentiating the last relation of (2.6) w.r.t. β and using $K_{\alpha,j} + \bar{K}_{j,\alpha} = 0 \Rightarrow K_{\alpha,\beta,j} = K_{\alpha,j\beta} = -\bar{K}_{j,\alpha\beta}$ we obtain:

$$\bar{K}_{j,\alpha\beta} + \Gamma_{\alpha\beta}^\gamma\bar{K}_{j,\gamma} = -\psi_{,j}g_{\alpha\beta} \tag{2.8}$$

where in the above equation the index j should be understood as counting the components of $\bar{K}_j$. With this convention we rewrite (2.8) (and a similar relation for K_α) as follows:

$$\bar{K}_{i|\alpha\beta} = -\psi_{,i}g_{\alpha\beta}, \qquad K_{\alpha||ij} = -\psi_{,\alpha}g_{ij}. \tag{2.9}$$

These equations mean that each $K_\alpha, \bar{K}_j$ (and for each value of the corresponding covariant index) is a gradient CVF of the metrics $g_{ij}, g_{\alpha\beta}$. Moreover Ricci's identity for the vector fields $\bar{K}_{i|\alpha}, K_{\alpha||i}$ implies:

$$(d-1)\,\psi_{,i\gamma} = R_{\delta\gamma}(\bar{K}_i)^{|\delta}, \qquad (m-1)\,\psi_{,i\gamma} = \bar{R}_{id}(K_\gamma)^{||d}. \tag{2.10}$$

where $R_{\alpha\beta}, \bar{R}_{ij}$ are the Ricci tensors associated with the constituent metrics $g_{\alpha\beta}, g_{ij}$ respectively.

We consider cases according to the type of the conformal symmetry.

KVFs

Then $\psi_{,A} = 0$ and equations (2.9), due to (2.6), imply that each pair $\mathbf{K} = \{K_{\alpha,i}, \bar{K}_{i,\alpha}\}$ must vanish in which case $K_\alpha(x^\beta)$ and $\bar{K}_i(x^j)$. We conclude that the KVFs of the $(d+m)-$decomposable metric g_{AB} *are identical* with the KVFs $K_\alpha(x^\beta)$ and $\bar{K}_i(x^j)$ of the constituent metrics.

HVFs

Since again $\psi_{,A} = 0$ the same arguements apply, therefore we conclude that the full decomposable metric admits a HVF H_A *if and only if* both the constituent metrics admit one, say $H_\alpha(x^\beta), \bar{H}_i(x^j)$. If ψ_1, ψ_2 are the (constant) conformal factors of the HVFs $H_\alpha(x^\beta), \bar{H}_i(x^j)$ respectively, then:

$$H_A = \psi_2 H_\alpha \delta_A^\alpha + \psi_1 \bar{H}_i \delta_A^i \tag{2.11}$$

with conformal factor $\psi = \psi_1\psi_2$.

Proper CVFs

In this case differentiating equations (2.10) w.r.t. "j" and "β" we obtain:

$$\psi_{|\alpha\beta} = -\frac{\bar{\Box}\psi}{m}g_{\alpha\beta}, \qquad \psi_{||ij} = -\frac{\Box\psi}{d}g_{ij} \tag{2.12}$$

where $\Box, \bar{\Box}$ denote the d'Alambertian operator w.r.t. the metrics $g_{\alpha\beta}, g_{ij}$ respectively. Therefore each of the constituent metrics admits the gradient CVFs $\psi_{,\alpha}, \psi_{,i}$. Furthermore the conformal factors $\bar{\Box}\psi, \Box\psi$ satisfy:

$$m\Box\psi + d\bar{\Box}\psi = 0 \tag{2.13}$$

We note that equations (2.12) are *necessary* (but not sufficient) conditions and can be used as a simple criterion to check whether a specific $(d+m)-$decomposable metric admits a proper CVF or not. However when the constituent metrics possess additional restrictions (e.g. they are Einstein spaces or spaces of constant curvature) we can draw further conclusions regarding the form of the CVFs and the structure of the metrics $g_{\alpha\beta}$, g_{ij} similarly to the case of four dimensional decomposable manifolds (see e.g. [3, 5] for details). □

3. An application

In brane cosmology we assume that our Universe is a defect embedded in a higher-dimensional bulk space [6] in which the bulk (warped) metric, in Gauss normal coordinates, can be written as $(A, B, ... = 0, 1, 2, 3, 4)$:

$$ds^2 = -m^2(\tau, \eta)dt^2 + d\eta^2 + R^2(\tau, \eta)d\Omega_k^2 \tag{3.1}$$

where $d\Omega_k$ is the three dimensional metric of constant curvature parametrized with $k = 0, \pm 1$.

The above approach has been provide a context in which the cosmological evolution can display novel features [7, 8]. Although the brane evolution depends on the matter content which is

assumed to be located in the bulk, the geometry of the brane is unaffected and is described by the existence of a G_6 group of isometries acting on the three dimensional spacelike orbits $\mathcal{S}_3$ of constant curvature. However the bulk geometry depends crucially on the dynamic assumptions we made for the matter content of the bulk. For example an extensively studied case in the recent literature is when $R^A_B = -\frac{2}{3}\Lambda\delta^A_B$ which results as its *unique solution*, the well known five-dimensional AdS-Schwarzschild static metric for the bulk [9].

This conclusion can be proved straightforward by means of Theorem 1 leading to:

Proposition 1. *Let $\mathcal{M}$ a $n-$dimensional pseudo-Riemannian manifold endowed with Lorentzian metric $\hat{\mathbf{g}}$ admitting a G_r group of isometries acting on $d-$dimensional spacelike orbits $\mathcal{S}_d$ with $r = \frac{d(d+1)}{2}$ ($\Leftrightarrow \mathcal{S}_d$ are of constant curvature and $d = n - 2$). Then a $\Lambda-$term or a degenerated algebraic type $[(1,1),1,...1]$ for the Ricci tensor implies that the metric $\hat{\mathbf{g}}$ admits a hypersurface orthogonal KVF thus a G_{r+1} group of isometries.*

The above proposition can be seen as the generalisation to $2 + (n - 2)$ warped geometries of Birkhoff's theorem (a four dimensional version of this theorem has been shown by Barnes and Goenner [11] using, however, a different method).

Consequently one can choose to study the brane evolution in the coordinate system adapted to the cosmological fuid observers u^A (metric (3.1)) in which the brane is located at a fixed value ($\eta = 0$) of the extra spatial coordinate [8, 10]. In this case the KVF is:

$$X^A = m^{-2}\left[b^2(R)\left(R_{,\tau}\right)^2 + m^2\right]^{1/2} n(R)\partial_\tau + m^{-1}b(R)n(R)R_{,\tau}\partial_\eta \tag{3.2}$$

with $b(R), n(R)$ arbitrary functions of the "scale factor" $R(\tau, \eta)$.

On the other hand we may choose a coordinate system adapted to the "fluid" velocity parallel to the KVF $X^A = \delta^A_t$ in which the brane is moving across the spatial direction [12, 13]. Then Proposition 1 shows that the two approaches are (geometrically) equivalent *only* for the case of a (negative) cosmological constant.

Although the assumption of $\Lambda-$term necessary implies the existence of a timelike Killing Vector Field (KVF), in the presence of bulk matter $T^A_D \neq \Lambda\delta^A_D$ the cosmological evolution on the brane is not autonomous, and the explicit knowledge of the bulk energy-momentum tensor is necessary. Moreover the bulk metric does not necessary admits an additional isometry. Therefore it is difficult to determine an exact solution towards an effective study of the possibility of energy exchange between the brane and the bulk. Geometrically this energy exchange is described by the energy flux vector field $q^A = -h^A_D u^B T^D_B$ (where $h^A_D = \delta^A_D + u^A u_D$) and can be used in order to study the induced modifications of the cosmological evolution on the brane. As a result, certain additional assumptions have been made for its form [15] or for the bulk geometry. For example one may relax the dynamical assumption of a cosmological constant for the bulk but maintain the existence of a *hypersurface orthogonal* timelike KVF $X^A = \delta^A_t$ for the bulk warped metric [16]. According to Theorem 1, this additional hypersurface KVF is actually a KVF of the constituent $2-$metric therefore the bulk metric can be written in the static form:

$$ds_5^2 = -n^2(r)dt^2 + b^2(r)dr^2 + r^2 d\Omega_k^2 \tag{3.3}$$

where $n(r), b(r)$ are arbitrary functions.

Employing a *perfect fluid* (plus a negative cosmological constant) for the matter content in the bulk with arbitrary equation of state, the fluid velocity u^A is parallel to $X^A = \delta^A_t$. In this case the most general solution for the metric (3.3) is:

$$\frac{1}{b^2} = k + \frac{1}{12M^3}\Lambda r^2 - \frac{1}{6\pi^2 M^3}\frac{1}{r^2}\mathcal{M}(r), \tag{3.4}$$

where $\mathcal{M}(r)$ satisfies

$$\frac{d\mathcal{M}}{dr} = 2\pi^2 r^3 \rho \qquad (3.5)$$

where ρ is the energy density of the perfect fluid as measured by the bulk observers u^A and the pressure satisfies the conservation equation:

$$\frac{p'}{\rho + p} = \frac{1}{r} - \frac{1}{r}\left[k + \frac{1}{6M^3}r^2(p + \Lambda)\right]\left[k + \frac{1}{12M^3}\Lambda r^2 - \frac{1}{6\pi^2 M^3}\frac{1}{r^2}\mathcal{M}(r)\right]^{-1} . \qquad (3.6)$$

We note that for $\rho = 0$ and constant $\mathcal{M}$ we obtain the known AdS-Schwarzschild solution [13]. Therefore the above solution can be interpreted as AdS-stars, as their form generalizes the known four-dimensional model for the interior of stellar configurations [14]. Furthermore for a complete solution an equation of state $p = p(\rho)$ must be provided. For example assuming a generic polytropic equation of state $p = w\rho^\gamma$ the large-r behaviour can be obtained analytically through the assumption that the cosmological constant term $\sim \Lambda r^2$ dominates over the curvature and matter terms. In this way we obtain

$$\rho(r) = \left(\kappa\, r^{\frac{1-\gamma}{\gamma}} - \frac{1}{w}\right)^{\frac{1}{\gamma-1}} \qquad \text{for } \gamma \neq 1 \qquad (3.7)$$

$$\rho(r) = \kappa\, r^{-\frac{w+1}{w}} \qquad \text{for } \gamma = 1 \qquad (3.8)$$

with κ an integration constant.

Clearly, under these assumptions the cosmological fluid observers $\tilde{u}^A = \delta^A_\tau$ measure an energy exchange q^A and although the explicit solution for the bulk is derived on the comoving coordinate system of the bulk fluid velocity u^A (or equivalently of the KVF X^A), the evolution equations for the brane can be studied in the standard coordinate system adapted to the brane obervers $\tilde{u}^A$.

The Hubble expansion of the brane becomes [16]:

$$\frac{\dot{R}^2}{R^2} = H^2 = \frac{1}{144M^6}\left(\tilde{\rho}^2 + 2V\tilde{\rho}\right) - \frac{k}{R^2} + \frac{1}{6\pi^2 M^3}\frac{\mathcal{M}(R)}{R^4} + \lambda. \qquad (3.9)$$

where $\tilde{\rho}$ is the energy density on the brane, V is the brane tension and λ is the effective cosmological constant. The novel feature of eq. (3.9) is the term $\sim \mathcal{M}(R)/R^4$. It is a generalization of the "mirage", or "Weyl", or "dark" radiation term [13] coming from the *energy outflow* from the brane, which is related to the mass $\mathcal{M}(R)$ of the AdS-star in the bulk.

Acknowledgments

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