THE ASTROPHYSICAL JOURNAL, 375:663–673, 1991 July 10

THE EVOLUTION OF DISCONTINUITIES IN RADIATING SPHERES IN THE DIFFUSION APPROXIMATION

W. BARRETO[1]
Laboratorio de física teórica, Departamento de física, Núcleo de Sucre, Universidad de Oriente, Cumana-Venezuela
L. HERRERA[2]
Departamento de física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas-Venezuela
AND
L. NÚÑEZ[3]
Laboratorio de física teórica, Departamento de física, Facultad de Ciencias, Universidad de los Andes, Mérida-Venezuela
Received 1990 August 22; accepted 1991 January 2

ABSTRACT

The evolution of discontinuities in a general relativistic sphere free of singularities is studied. The energy transport mechanism through fluid is diffusive. The distribution of matter is divided by a shock wave front in two regions. The equations of state at both sides of the shock are different, and the solutions are matched on it via the Rankine-Hugoniot conditions. The outer metric joins the Vaidya solution at the boundary surface of the sphere. Exploding models are obtained, and their dynamics are studied using a generalized compressibility coefficient for nonadiabatic systems.

Subject headings: diffusion — relativity — shock waves

1. INTRODUCTION

In the general relativistic studies on the evolution of fluid spheres, dissipative processes are generally absent. However, in certain situations of stellar collapse, those processes might play an important role. For example, the opacity of collapsing matter for neutrinos and/or photons will give rise to diffusion processes associated with radiative thermal conduction (Kazanas 1978; Mihalas & Mihalas 1984). Also, as it has been recently reported, the low mean energy of the detected neutrinos (≈ 10 MeV) from the supernova 1987a and the long duration of the burst attest to the diffusion, not streaming out, of the neutrinos from the lepton-rich neutron star (Lattimer 1988).

In this paper we shall propose an extension of a method reported by Herrera & Núñez (1987) to study the evolution of discontinuities in radiating spherically symmetric distribution of matter. In the present case the energy transport scheme is diffusive in the two regions of the sphere divided by a shock wave front. We do not discuss either the mechanism for inducing diffusion, or the corresponding underlying microphysics, rather we concentrate on the influence of diffusion processes on the evolution of radiating self gravitating spheres. At either side of the shock wave front, a physically reasonable equation of state is chosen. The solutions are matched across it via the Rankine-Hugoniot conditions (Taub 1948, 1983; Herrera & Núñez 1987). The matching with the Vaidya metric on the boundary of the configuration completes the consistency of the radiating sphere. These junction conditions at the shock and at the boundary of the matter distribution, lead to a system of ordinary differential equations for quantities evaluated at either the boundary surface or the shock front. The numerical integration of this system allow us, using the fields equations, to find the profile of the physical variables throughout. In order to understand further the dynamics of the system, we shall use a generalization of the concept of adiabatic index to nonadiabatic situations. This essentially gives a measure of the stiffening of any piece of material during the evolution (Barreto, Herrera, & Santos 1990). As we shall see, the introduction of this parameter allows us to set up a self-consistent explanation of the bouncing of the outer regions without making appeal of a strong shock erupting at the boundary surface. In the context of our approach the role played by the discontinuity surface (the shock) is that of an interface separating two regions with different equations of state. After the emission of the pulse, this surface becomes the boundary of the (quasi-) homogeneous compact remnant (Herrera & Núñez 1990). The paper is organized as follows. In § 2 the conventions used, the field equations, the nonadiabatic index, and the method are sketched. Section 3 contains a model worked out. Finally the results are discussed in the last section.

2. RADIATING FLUID SPHERE WITH SHOCK AND HEAT FLOW

2.1. *Field Equations*

Let us consider a nonstatic distribution of matter which is spherically symmetric and consists of fluid unpolarized radiation traveling in the radial direction (to model the streaming-out regime), heat flow (to model diffusion regime), and an isotropic radiation. In radiation coordinates (Bondi 1964), the metric takes the form

$$ds^2 = e^{2\beta}\left(\frac{V}{r}\,du^2 + 2du\,dr\right) - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,, \quad (1)$$

where β and V are functions of u and r. Here $u \equiv x^0$ is a timelike coordinate. In flat space-time u is just the retarded time, therefore surfaces $u =$ constant represent null cones open to the future. $r \equiv x^1$ is a null coordinate ($g_{rr} = 0$) such that surface $r =$ constant, $u =$ constant are spheres, and θ, $\phi = x^2$, x^3 are the usual angle coordinates. The relationship between these coordinates and the usual Schwarzschild coordinates

[1] Postal address: Apartado 42, Cumaná, Venezuela.
[2] Postal address: Apartado 80793, Caracas 1080A, Venezuela.
[3] Postal address: Apartado 54, Mérida, Venezuela.

$(T, R, \Theta. \Phi)$ is given by

$$u = T - \int \frac{r}{V}\, dr\,, \quad \Theta = \theta\,, \qquad r = R\,, \quad \Phi = \phi\,. \tag{2}$$

For the matter distribution considered here, the energy-momentum tensor has the form

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - Pg_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \epsilon k_\mu k_\nu\,, \tag{3}$$

where u_μ, q_μ and k_μ denote, respectively, the four-velocity of the fluid, the heat flux vector satisfying the condition

$$q^\mu u_\mu = 0 \tag{4}$$

and a null vector pointing in the direction of the outgoing radiation. Now, in order to give a physical meaning to the components of the energy-momentum tensor, as given by equation (3) let us (following Bondi 1964) introduce purely local Minkowski coordinates (t, x, y, z) by

$$\begin{aligned} dt &= e^\beta[(V/r)^{1/2}\, du + (r/V)^{1/2}\, dr]\,, \\ dx &= e^\beta(r/V)^{1/2}\, dr\,, \\ dy &= r\, d\theta\,, \\ dz &= r \sin\theta\, d\phi\,. \end{aligned} \tag{5}$$

Then, denoting Minkowskian components of the energy-momentum tensor by a caret, we have

$$\begin{aligned} T_{00} &= \hat{T}_{00}\left(e^{2\beta}\frac{V}{r}\right), \\ T_{01} &= (\hat{T}_{00} + \hat{T}_{01})e^{2\beta}\,, \\ T_{11} &= (\hat{T}_{00} + \hat{T}_{11} + 2\hat{T}_{01})e^{2\beta}\frac{r}{V}\,, \\ T^2_2 &= T^3_3 = \hat{T}^2_2 = \hat{T}^3_3\,. \end{aligned} \tag{6}$$

Next, one assumes that for an observer moving relative to these coordinates with velocity ω in the radial direction, the space contains

1. An isotropic fluid of density $\hat{\rho}$ and pressure $\hat{P}$.
2. Unpolarized radiation of energy density $\hat{\epsilon}$ traveling in the radial direction.
3. A radial heat flux $\hat{q}$.
4. Isotropic radiation of energy density 3 $\hat{\sigma}$.

For this specific observer, the covariant energy tensor is

$$\begin{pmatrix} \hat{\rho} + 3\hat{\sigma} + \hat{\epsilon} & -\hat{q} - \hat{\epsilon} & 0 & 0 \\ -\hat{q} - \hat{\epsilon} & \hat{\rho} + \hat{\sigma} + \hat{\epsilon} & 0 & 0 \\ 0 & 0 & \hat{\rho} + \hat{\sigma} & 0 \\ 0 & 0 & 0 & \hat{\rho} + \hat{\sigma} \end{pmatrix}, \tag{7}$$

and a unit spacelike vector radially oriented (along the x-coordinate) has components

$$(0, -1, 0, 0)\,. \tag{8}$$

Then a Lorentz transformation readily shows that

$$\hat{T}^{00} = \hat{T}_{00} = \frac{\rho + P\omega^2}{1 - \omega^2} + \frac{2Q\omega}{(1 - \omega^2)^{1/2}(1 - 2\tilde{m}/r)^{1/2}} + \epsilon\,, \tag{9}$$

$$\hat{T}^{11} = \hat{T}_{11} = \frac{P + \rho\omega^2}{1 - \omega^2} + \frac{2Q\omega}{(1 - \omega^2)^{1/2}(1 - 2\tilde{m}/r)^{1/2}} + \epsilon\,, \tag{10}$$

$$-\hat{T}^{01} = \hat{T}_{01} = -\frac{\omega(\rho + P)}{(1 - \omega^2)} - \frac{Q(1 + \omega^2)}{(1 - \omega^2)^{1/2}(1 - 2\tilde{m}/r)^{1/2}} - \epsilon\,, \tag{11}$$

$$\hat{T}^2_2 = \hat{T}^3_3 = -P\,, \tag{12}$$

where

$$\rho = \hat{\rho} + 3\hat{\sigma}\,, \qquad P = \hat{P} + \hat{\sigma}\,, \qquad \epsilon = \hat{\epsilon}\left(\frac{1 + \omega}{1 - \omega}\right), \qquad V = e^{2\beta}(r - 2\tilde{m})\,, \tag{13}$$

and

$$Q = \hat{q}\,\frac{(1 - 2\tilde{m}/r)^{1/2}}{(1 - \omega^2)^{1/2}}\,, \tag{14}$$

and for the unit radial vector we get

$$\hat{n}^0 = \hat{n}_0 = \frac{\omega}{(1 - \omega^2)^{1/2}}\,, \tag{15}$$

$$\hat{n}^1 = -\hat{n}_1 = \frac{-1}{(1 - \omega^2)^{1/2}}\,. \tag{16}$$

From equation (5) above it follows that the velocity of a given fluid element in the null coordinate system is given by

$$\frac{dr}{du} = \frac{V}{r}\frac{\omega}{1 - \omega}\,. \tag{17}$$

Also, it is worth noticing that the function $\tilde{m}(u, r)$ is the mass in the static case, and its value in the boundary surface of the sphere equals the Vaidya mass (Bondi 1964). So far we have considered the general case when both the heat flow term and the outgoing null radiation are present. In what follows, we shall restrict ourselves to the case we are concerned with in this paper. Thus we shall assume $\epsilon = 0$. The purely streaming-out case ($q = 0$, $\epsilon \neq 0$) has been considered before (Herrera & Núñez 1987, 1990)

Then the Einstein fields equations may be written in radiative (null) coordinates as

$$\frac{\rho + P\omega^2}{1 - \omega^2} + \frac{2Q\omega}{(1 - \omega^2)^{1/2}(1 - 2\tilde{m}/r)^{1/2}} = \frac{1}{4\pi r(r - 2\tilde{m})}\left(-\tilde{m}_0 e^{-2\beta} + \frac{r - 2\tilde{m}}{r}\tilde{m}_1\right), \tag{18}$$

$$\frac{\rho - P\omega}{1 + \omega} - \frac{(1 - \omega)Q}{(1 - 2\tilde{m}/r)^{1/2}}\left[\frac{1 - \omega}{1 + \omega}\right]^{1/2} = \frac{\tilde{m}_1}{4\pi r^2}\,, \tag{19}$$

$$\frac{1 - \omega}{1 + \omega}(\rho + P) - 2\frac{(1 - \omega)Q}{(1 - 2\tilde{m}/r)^{1/2}}\left[\frac{1 - \omega}{1 + \omega}\right]^{1/2} = \frac{r - 2\tilde{m}}{2\pi r^2}\beta_1\,, \tag{20}$$

$$P = -\frac{\beta_{01}e^{-2\beta}}{4\pi} + \frac{1}{8\pi}\left(1 - \frac{2\tilde{m}}{r}\right) \times \left(2\beta_{11} + 4\beta_1^2 - \frac{\beta_1}{r}\right) + \frac{3\beta_1(1 - 2\tilde{m}_1) - \tilde{m}_{11}}{8\pi r}\,. \tag{21}$$

Here, differentiation with respect to u and r is denoted by subscripts 0 and 1, respectively.

We consider the sphere of radius $a(u)$ divided in two regions (denoted by I and II) by a discontinuity surface (a shock) at a radius $c(u)$. The matching conditions across the shock (the Rankine-Hugoniot conditions) require the continuity of the first and second fundamental forms, plus the continuity of $T_{\mu\nu} n^\nu$, where $T_{\mu\nu}$ is the energy-momentum tensor and n^ν is the unit vector normal to the surface $r = c(u)$. The continuity of the first and second forms leads to the following conditions (Taub 1948, 1983; Herrera & Núñez 1987):

$$[\beta]_c = [\tilde{m}]_c = 0 , \tag{22}$$

and

$$\left[2\beta_1 e^{2\beta}\left(1 - \frac{2\tilde{m}}{r}\right) - 2\beta_0 - e^{2\beta}\,\frac{\tilde{m}_1}{r}\right]_c = 0 , \tag{23}$$

where

$$[f]_c = 0 \equiv f|_{r=c+0} - f|_{r=c+0} = f_{\rm II} - f_{\rm I} .$$

Next, since β is continuous across $c(u)$,

$$\beta(r, u) \approx \beta(c, u) + \beta_1|_c(r - c) ,$$

and then

$$[\beta_0 + c_0 \beta_1]_c = 0 . \tag{24}$$

Substituting equation (24) back into equation (23) and using equations (19) and (20), we get

$$[(\tilde{\rho} + \tilde{P})\Psi - \tilde{\rho}]_c = 0 , \tag{25}$$

with (see eqs. [39], [40] below)

$$\Psi \equiv 1 + c_0 e^{-2\beta}(1 - 2\tilde{m}/r)^{-1} . \tag{26}$$

The conditions $[T_{\mu\nu} n^\nu]_c = 0$ read

$$[(\tilde{\rho} + \tilde{P})\Psi - \tilde{\rho}]_c = 0 , \tag{27}$$

which is exactly the condition (25) (Herrera et al. 1987), and

$$\left[\frac{\rho + \omega^2 P}{1 - \omega^2} - \Psi\,\frac{\rho - \omega P}{1 + \omega} + \frac{Q[2\omega + \Psi(1 - \omega^2)]}{(1 - 2\tilde{m}/r)^{1/2}(1 - \omega_2)^{1/2}}\right]_c = 0 . \tag{28}$$

At this point the following remark is in order: since $r = c(u)$ is a surface of discontinuity, then

$$\frac{dc}{du} = c_0 \neq v|_{r=c} = \left[\frac{dr}{du}\right]_c .$$

In other words, the velocity of the shock, in general, is not equal to the velocity of matter evaluated at $r = c(u)$. Should $r = c(u)$ not be a discontinuity surface, then

$$c_0 = v|_{r=c} = \left[\frac{dr}{du}\right]_c ,$$

and it is easily found from equations (17) and (27) that

$$\Psi = \frac{1}{1 - \omega} . \tag{29}$$

Substituting equation (29) back into equation (27) we get

$$\left[P - \frac{Q(1 - \omega^2)^{1/2}}{(1 - 2\tilde{m}/r)^{1/2}}\right]_{r=c} = 0 , \tag{30}$$

which indicates that the hydrodynamic pressure is not continuous across $r = c(u)$ as expected for a dissipative fluid (Santos 1985; Herrera et al. 1987; Lake 1987). This is the situation at the boundary surface $r = a(u)$, i.e., for a surface to be a boundary, the velocity of matter must be equal to the velocity of the surface itself.

2.2. *Generalized Concept of Adiabatic Index for Nonadiabatic Systems*

To study the evolution of the radiating system we include, briefly, in this part a definition of compressibility proposed by Barreto et al. (1990). This concept takes into account the emission and absorption processes and is obtained by extrapolating the physical meaning of the terms appearing in the definition of the adiabatic index, that is,

$$\Gamma = \frac{d \ln P}{d \ln \rho} , \tag{31}$$

where P and ρ are the hydrodynamic pressure and energy density, respectively. The generalized concept of adiabatic index is given by

$$\Gamma_{\rm N.A.} = \frac{d \ln \Pi}{d \ln E} , \tag{32}$$

where Π and E are the total flux of momentum in the direction of contraction and the total energy density as measured by some suitable defined Minkowskian observer (N.A. stands for nonadiabatic). This definition is completely general; it is not related to a specific degree of symmetry of the system or to a given radiation transport regime.

For our spherically symmetric system the total flux of momentum in the radial (x) direction, as measured by our (x, y, z, t) local Minkowski observer, is given by

$$\Pi = \hat{T}^{\alpha 1} \hat{n}_\alpha , \tag{33}$$

which we can obtain, using equations (15)–(16) and (9)–(12)

$$\Pi = \frac{2\omega^2 \rho + P(1 + \omega^2)}{(1 - \omega^2)^{3/2}} + \frac{Q\omega}{(1 - \omega^2)(1 - 2\tilde{m}/r)^{1/2}}\,(3 + \omega^2) . \tag{34}$$

Note that in the case of slow ($\omega^2 \approx 0$) and adiabatic ($Q = 0$) contraction, equation (34) reduces to hydrodynamic pressure. Then the generalized concept of hydrodynamic adiabatic index is given by

$$\Gamma_{\rm N.A.} = \frac{d \ln \Pi}{d \ln \hat{T}_{00}} . \tag{32'}$$

It may be calculated using equations (34) and (9). Observe that in the slow adiabatic contraction we get $\Gamma_{\rm N.A.} \to \Gamma$.

2.3. *The Method*

Using the fields equations (19) and (20), we obtain for the metric functions β and $\tilde{m}$

$$\tilde{m}_{\rm I} = \int_0^r 4\pi\tau^2 \tilde{\rho}_{\rm I}\, d\tau , \qquad 0 \le r \le c(u) , \tag{35}$$

$$\tilde{m}_{\rm II} = \int_{c(u)}^r 4\pi\tau^2 \tilde{\rho}_{\rm II}\, d\tau + \tilde{m}_{\rm I}[u, c(u)] , \qquad c(u) \le r \le a(u) , \tag{36}$$

and

$$\beta_{\rm I} = \int_{c(u)}^{r} \frac{2\pi\tau^2}{\tau - 2\tilde{m}_{\rm I}} (\tilde{\rho}_{\rm I} + \tilde{P}_{\rm I}) d\tau + \beta_{\rm II}[u, c(u)] , \quad 0 \le r \le a(u) , \tag{37}$$

$$\beta_{\rm II} = \int_{a(u)}^{r} \frac{2\pi\tau^2}{\tau - 2\tilde{m}_{\rm II}} (\tilde{\rho}_{\rm II} + \tilde{P}_{\rm II}) d\tau , \quad c(u) \le r \le a(u) , \tag{38}$$

where we have defined the two auxiliary functions

$$\tilde{\rho} \equiv \frac{\rho - \omega P}{1 + \omega} - \frac{(1-\omega)Q}{(1 - 2\tilde{m}/r)^{1/2}} \left[\frac{1-\omega}{1+\omega}\right]^{1/2} , \tag{39}$$

$$\tilde{P} \equiv \frac{P - \omega \rho}{1 + \omega} - \frac{(1-\omega)Q}{(1 - 2\tilde{m}/r)^{1/2}} \left[\frac{1-\omega}{1+\omega}\right]^{1/2} , \tag{40}$$

hereafter referred to as effective density and effective pressure, respectively. The subscripts I and II indicate the region where the quantity is evaluated. We can now restate the Herrera & Núñez algorithm (1987) for the composite sphere with heat flow:

1. Take two static interior solutions of the Einstein equations for distributions of matter with spherical symmetry, given

$$\rho_{\rm stI} = \rho_{\rm I}(r) \ ; \quad \rho_{\rm stII} = \rho_{\rm II}(r) ,$$
$$P_{\rm stI} = P_{\rm I}(r) \ ; \quad P_{\rm stII} = P_{\rm II}(r) .$$

2. Assume that the r-dependence of $\tilde{P}_{\rm I,II}$ and $\tilde{\rho}_{\rm I,II}$ is the same as that of $P_{\rm stI,II}$ and $\rho_{\rm stI,II}$, but taking care of the boundary condition which, for this case, reads

$$\tilde{P}_{{\rm II}a} = -\omega_a \tilde{\rho}_{{\rm II}a} ,$$

and the conditions at $r = c(u)$ given by equations (27) and (28).

3. With the r-dependence of $\tilde{\rho}_{\rm I,II}$ and $\tilde{P}_{\rm I,II}$, and using equations (35)–(38), one gets $\tilde{m}_{\rm I,II}$ and $\beta_{\rm I,II}$ up to some functions of u which will be specified below.

4. For these functions of u, one obtains the following ordinary differential equations (surface equations). Three of them (two model dependent) emerge because of the junction conditions and the field equations evaluated at the boundary surface. The remainder of the surface equations comes from the Rankine-Hugoniot conditions.

5. Providing three additional functions, one evaluated at $r = a(u)$ and the other two evaluated at $c(u)$, the system of the superficial equations may be integrated for any particular set of initial data.

6. Feeding back the result of integration in the expressions for $\tilde{m}_{\rm I,II}$ and $\beta_{\rm I,II}$, these four functions become completely determined.

2.4. *Surface Equations*

As it should be clear, the crucial point in the above algorithm is the system of surface equations, described as follows:

1. The first surface equation comes from equation (17) evaluated at the boundary $r = a(u)$, which after scaling the radius a, the total mass m, and the timelike coordinate u by the initial mass $m(u = 0) = m(0)$

$$A = a/m(0) , \quad M = m/m(0) , \quad u/m(0) \to u ,$$

and defining

$$F \equiv 1 - \frac{2M}{A} , \quad \Omega \equiv \frac{1}{1 - \omega_a} ,$$

can be written as

$$\dot{A} = F(\Omega - 1) . \tag{41}$$

2. The second surface equation relates the total loss rate with the energy flux through the boundary surface. This can be obtained by evaluating equation (18) at $r = a \pm 0$, and we get (Herrera et al. 1987)

$$\dot{F} = \frac{[2F^{1/2}q(2\Omega - 1)^{3/2}/\Omega] + \dot{A}(1 - F)}{A} , \tag{42}$$

with $q = 4\pi a^2 Q_a$.

3. The third equation at the boundary surface will be obtained from the conservation equation $T^{\mu}_{1;\mu} = 0$ evaluated at the surface.

4. The remaining equations are also model-dependent and are written as a result of the matching conditions across $r = c(u)$.

3. A MODEL

In this section we shall explicitly work out a model to illustrate the method just presented above. Let us first specify the effective variables in both regions of the sphere. For the inner region, a solution having as its static limit the well-known Schwarzschild homogeneous interior solution has been considered:

$$\tilde{\rho}_{\rm I} = f(u) , \tag{43}$$

$$\tilde{P}_{\rm I} = f(u) \left\{ \frac{g(u)[1 - (8\pi/3)r^2 f(u)]^{1/2} - 1/3}{1 - g(u)[1 - (8\pi/3)r^2 f(u)]^{1/2}} \right\} , \tag{44}$$

where $f(u)$ and $g(u)$ are two arbitrary functions of u. We can now integrate equations (35) and (37) to obtain

$$\tilde{m}_{\rm I}(u, r) = \frac{4\pi}{3} f(u) r^3 , \tag{45}$$

$$\beta_{\rm I} = -\frac{1}{2} \ln \frac{w(r)[g(u)w(c) - 1]}{w(c)[g(u)w(r) - 1]} + \beta_{\rm II}[u, c(u)] , \tag{46}$$

where

$$w(x) = [1 - 8\pi x^2(u) f(u)/3]^{1/2} .$$

So far we have three unknown functions of u which are $f(u)$, $g(u)$ and $\beta_{\rm II}[u, c(u)]$. For the outer region, the effective variables are chosen inspired in the static Tolman VI metric (Tolman 1939):

$$\tilde{\rho}_{\rm II} = \frac{3h(u)}{r^2} , \tag{47}$$

$$\tilde{P}_{\rm II} = \frac{h(u)}{r^2} \left[\frac{1 - 9k(u)r}{1 - k(u)r}\right] , \tag{48}$$

where $h(u)$ and $k(u)$ are unknown functions of u, related through the boundary condition $\tilde{P}_{{\rm II}a} = -\omega_a \tilde{\rho}_{{\rm II}a}$. Thus, we get for the metric functions

$$\tilde{m}_{\rm II}(u, r) = 12\pi h(u) r - 4\pi c(u)[3h(u) - c^2(u) f(u)/3] , \tag{49}$$

$$\beta_{\rm II}(u, r) = \frac{8\pi h(u)}{\gamma k(u) + \delta} \left\{ 2 \ln\left[\frac{1 - k(u)r}{1 - k(u)a(u)}\right] + \left[1 + \frac{3k(u)\gamma}{\delta}\right] \ln\left[\frac{\gamma + \delta r}{\gamma + \delta a(u)}\right] \right\} , \tag{50}$$

with

$$\gamma = 2[12\pi c(u)h(u) - 4\pi c^3(u)f(u)/3] \,,$$

and

$$\delta = 1 - 24\pi h(u) \,.$$

Having specified the solutions at either side of the shock (up to some functions of u), we are now ready to write out the surface equations. Two of them correspond to the surface at $r = a(u)$ (eqs. [41] and [42]). The third one can be obtained from the equation $[T^{\mu}_{1,\mu}]_{r=a} = 0$ after lengthy and tedious calculations. If the effective density is separable, i.e., $\tilde{\rho} = f(u)g(r)$, we get

$$\begin{aligned}\dot{\Omega} = \Omega\Bigg(&\frac{\dot{H}}{H} - 2\frac{\dot{A}}{A} - \frac{\dot{F}}{F} + \frac{(1+F)(\Omega-1)}{A} + \frac{A^2F\Omega}{3H} \\ &\times\left[\frac{2q}{[\Omega F(2\Omega-1)]^{1/2}4\pi A^3} + \frac{2(\Omega-1)^3H}{A^3\Omega} - R_a(u)\right] \\ &- F(\Omega-1)\Bigg[\!\Bigg[\frac{72\pi H^2}{\Omega A^3F^2} - \frac{1}{F} \\ &\times\left\{\frac{6H}{A^3} + \frac{H}{A^2(1-KA)}\left[\frac{8K}{1-KA} + \frac{2(1-9KA)}{A}\right]\right\}\Bigg]\!\Bigg]\Bigg) \,,\end{aligned} \quad (51)$$

where

$$\begin{aligned}R_a(u) = &-\frac{H}{A^2(1-KA)}\left[\frac{8K}{1-KA} + \frac{2(1-9KA)}{A}\right] \\ &+ \frac{3H}{A^3\Omega F}\left[\frac{1-F}{2} - \frac{12\pi H(\Omega-1)}{\Omega}\right] \,,\end{aligned} \quad (52)$$

with

$$H = \frac{A(1-F) - C(1-G)}{24\pi(A-C)} \,,$$

$$K = \frac{4\Omega - 3}{3A(4\Omega - 1)} \,,$$

and

$$C \equiv c(u)/m(0) \,, \quad M \equiv m_{1c}/m(0) \,, \quad G \equiv 1 - 2M/C \,.$$

Observe that the equations (41), (42), and (51) are coupled with the solutions at $r = c(u)$ through the appearance of the function $f(u)$ in equations (49) and (50).

For the fourth equation, we can use the condition (27), taking into account equations (45), (46), (49), and (50). Thus

$$\begin{aligned}\dot{C} = -\frac{G}{D}\,[\![&\{3D(1-KC)[(1-G)(3A-2C) - (1-F)A]\} \\ &\times\{2(1-G)[3(1-KC)(A-C) - 2DC(1-3KC)] \\ &+ 4(1-F)AD(1-3KC)\}^{-1} + 1]\!] \,,\end{aligned} \quad (53)$$

where

$$\Delta = \frac{\zeta^2}{\xi^2} \,, \quad \zeta = \phi^{\Lambda} \,, \quad \xi = \theta^{\Gamma}$$

with

$$\phi = \frac{\gamma + \delta C}{\gamma + \delta A} \,, \quad \Lambda = -\frac{8\pi H}{\delta}\left(\frac{\delta + 3K\gamma}{\gamma K + \delta}\right) \,,$$

$$\theta = \frac{1 - KC}{1 - KA} \,, \quad \Gamma = \frac{16\pi H}{\gamma K + \delta} \,,$$

and

$$D = gG^{1/2} - 1 \,.$$

We can use, equivalently, the field equations (18)–(21) evaluated at each side of the shock, instead of the condition (28), to obtain the other surface equations

$$\chi\dot{G}^3 + \Phi\dot{G}^2 + \eta\dot{G} + \Sigma = 0 \quad (54)$$

that can be solved easily, algebraically in G, and

$$\dot{D} = \frac{\Upsilon}{\Theta} \quad (55)$$

the quantities χ, Φ, η, Σ, Υ, and Θ are very complex functions of A, F, Ω, C, G, D, and their derivatives (see Appendix). Thus we are left with six equations for nine unknown functions of u: A, F, C, Ω, D, G, S, $q_{r=c-0}$ and $q_{r=c+0}$. To fulfill the numerical integration of the system of surface equations (41), (42), (51), (53), (54), and (55), we provide as "observational" input the functions S, $q_{r=c-0}$, and $q_{r=c+0}$. Physically S represents the total radiated mass per unit of retarded time u; $q_{r=c-0}$ and $q_{r=c+0}$ are the energy flux density per unit of time u at both sides of the shock wave front. We have selected

$$S = \frac{qF^{1/2}(2\Omega - 1)^{3/2}}{\Omega} \,,$$

as a Gaussian pulse such that the total ejected mass is 1/10 of the initial total mass. At both sides of the shock we have also chosen a Gaussian pulse. The total radiated energy flux at $r = c(u)$ is 10^{-4} of the total initial mass $m(0)$. The peak of the pulse at the shock is emitted four units of time before the maximum of the luminosity at the boundary surface is signaled. The opacity (emissivity) of the shock front may be varied modifying the shape of the pulses at both sides. The effect of this parameter on the evolution of the discontinuity surface will be analyzed elsewhere.

The surface equations were integrated numerically with the following initial data:

$$A(0) = 20 \,, \quad C(0) = 12 \,,$$

$$F(0) = 0.9 \,, \quad G(0) = 0.907 \,,$$

$$\Omega(0) = 0.92 \,, \quad D(0) = -0.598 \,,$$

which emerges as the most interesting model from a large number of other data sets considered. Figure 1 shows the evolution of the boundary surface $A(u)$ and the shock at $C(u)$. As was stated before, the physical variables are obtained from equations (18)–(21). The evolution of these matter variables are monitored at fixed radius labeled $r/a(0)$ and are displayed in Figures 2 through 5. Figure 6*a* exhibits the march of the nonadiabatic index at three fixed-mass shells. Figure 6*b* represents the values of this index through the sphere at four different times. Finally, Figures 7 and 8 sketched the variation of the nonadiabatic index and the matter velocity, respectively, at both sides of the shock. The next section is devoted to the discussion of results.

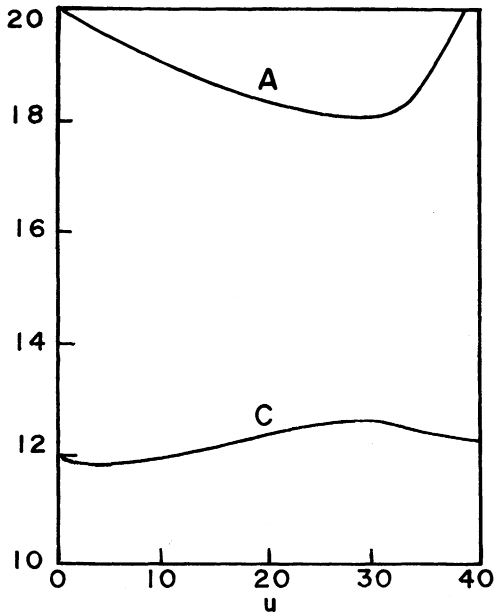

FIG. 1.—Evolution of $A(u)$ and $C(u)$

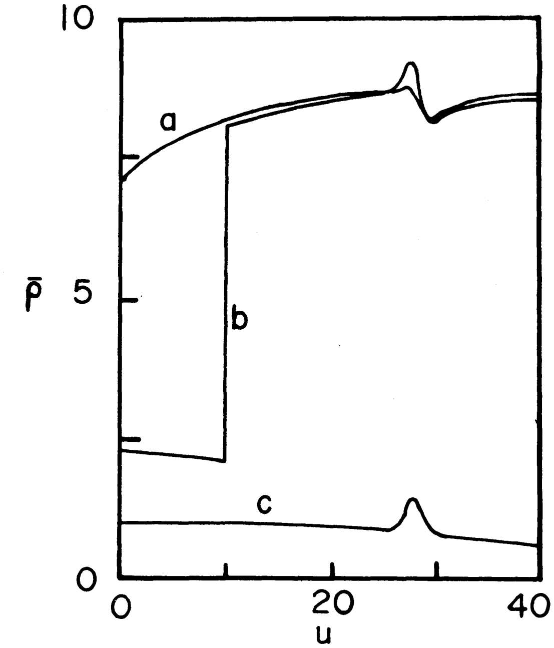

FIG. 3.—Evolution of the dimensionless density $\bar{\rho} \equiv \rho m(0)^2$ (multiplied by 10^5), monitored at different regions: curves a–c are for $r/a(0) = 0.2, 0.6$, and 1, respectively.

4. DISCUSSION OF THE RESULTS

The matter distribution considered is free of singularities everywhere. The Rankine-Hugoniot relations match, through the shock, the inner and the outer solutions, and the junction conditions across the boundary surface fulfill the remaining coupling with the radiating version of the Scharzschild exterior solution (the Vaidya solution). The diffusion limit is assumed through the composited sphere, except at the boundary surface and at both sides of the shock front. Figure 1 represents the evolution of the discontinuity and the boundary surfaces. It is clear from this figure that the shock does not reach the boundary of the matter distribution. Instead, it expands and later it recedes. During this period the outer boundary bounces and expands with an increasing velocity. It can be noticed that the expansion of the core enhances the expulsion of the outer mantle. This piston-like effect mechanism can be also observed in Figure 4 where the velocity profiles recorded at different mass shell are displayed. It is also apparent from this figure

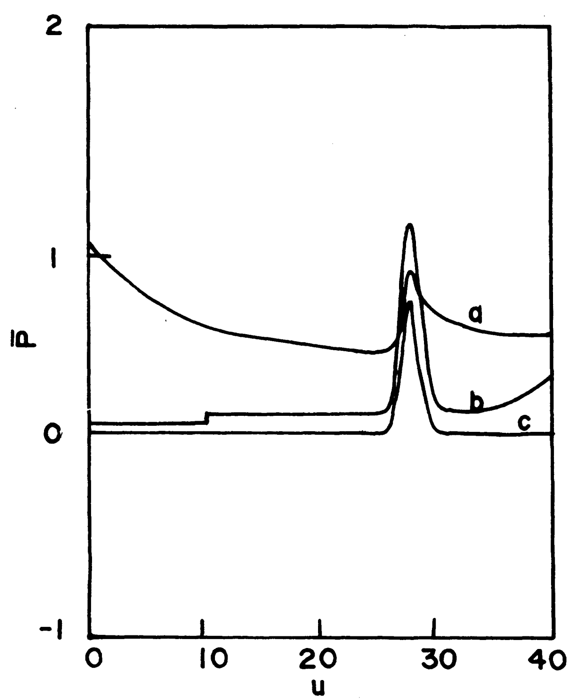

FIG. 2.—Evolution of the dimensionless pressure $\bar{P} \equiv Pm(0)^2$ (multiplied by 10^5), monitored at different regions: curves a–c are for $r/a(0) = 0.2, 0.6$ and 1, respectively.

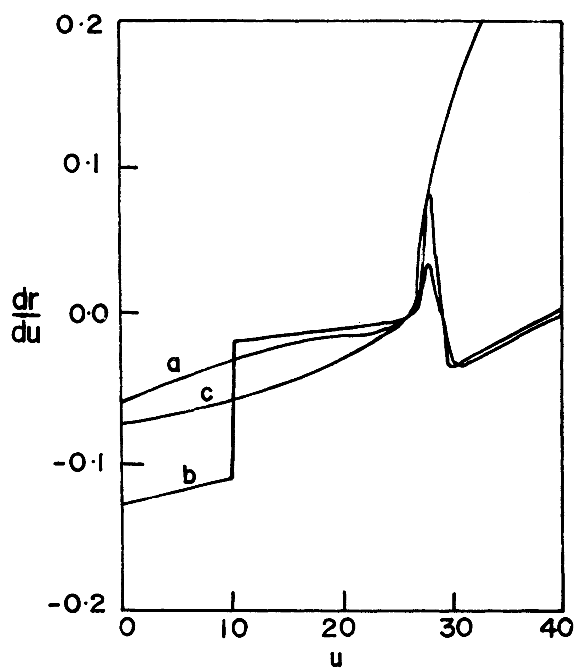

FIG. 4.—Evolution of the matter velocity (dr/du), monitored at different regions: curves a–c are for $r/a(0) = 0.2, 0.6$, and 1, respectively.

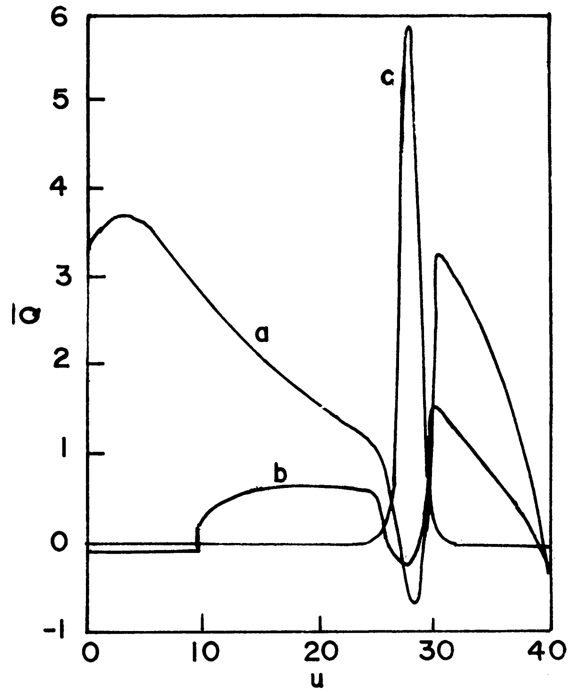

FIG. 5.—Evolution of the dimensionless heat flow $\bar{Q} \equiv Qm(0)^2$ (multiplied by 10^6), monitored at different regions: curves a–c are for $r/a(0) = 0.2, 0.6$, and 1, respectively.

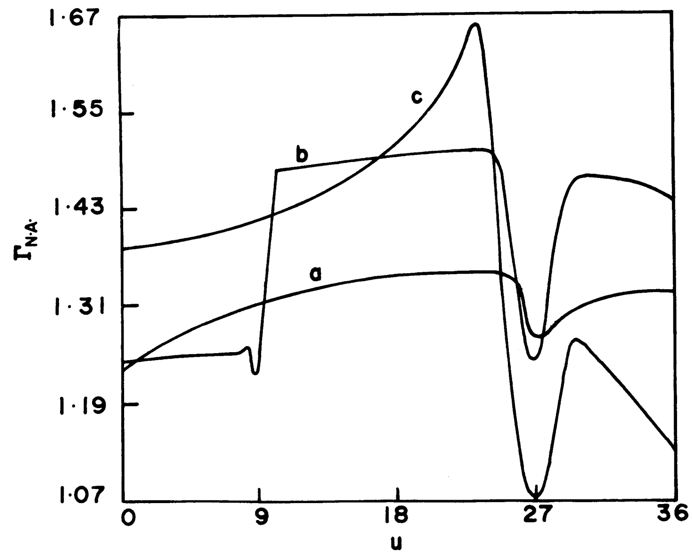

FIG. 6a

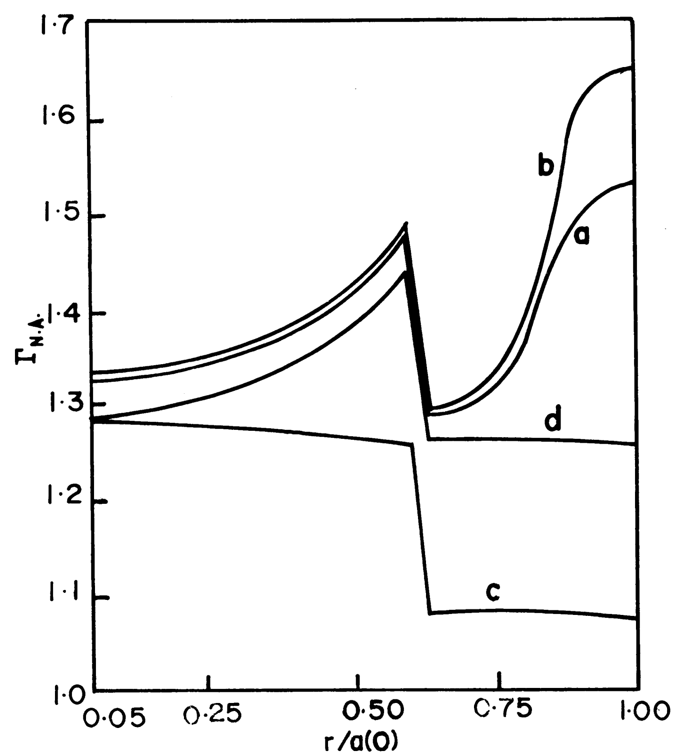

FIG. 6b

FIG. 6.—(*a*) Evolution of the nonadiabatic index, monitored at different regions: curves a–c are for $r/a(0) = 0.2$, 0.6, and 1, respectively. (*b*) Nonadiabatic index as a function of the dimensionless coordinate $r/a(0)$, monitored at different times: curves a–d are for $u = 20, 24, 28$, and 30, respectively.

that the bounce in every mass shell occurs at the same retarded time $u =$ constant. Now, as it follows from equation (2) for two events (*a*) and (*b*) happening at the same "retarded" time u, we have

$$T_a - T_b = \left(\int \frac{r}{V}\,dr\right)_a - \left(\int \frac{r}{V}\,dr\right)_b .$$

Since asymptotically (as $r \to \infty$) $V \approx r$ (Bondi Van der Burg, & Metzner 1962), then

$$T_a - T_b \approx r_a - r_b + 0 \text{ (gravitational potential)} .$$

Therefore inner shells bounce earlier than the outer ones, as seen by a distant observer. This result reveals a deep difference between the free streaming out and diffusion regimes (Herrera & Núñez 1987, 1990). The Gaussian-like shape of the pressure profile at the boundary surface is direct consequence of the junction condition given by equation (30). The stiffness of both, the inner core and the outer mantle as measured by the nonadiabatic index can be readily observed in Figures 6*a* and 6*b*. The nonadiabatic index of the material is greater at the inner side of the shock than at the mantle side. It is worth noticing that the emission of radiation diminishes the nonadiabatic index of the material softening the sphere. This effect is clear from Figure 6*a*. Moreover, after the emission of the radiation, the inner shells of the core asymptotically collapse to the hydrostatic limit. It is important to notice how well-defined zones (stiff core and soft mantle) emerge after the emission of the radiation (curves a and d, Fig. 6*b*).

Let us now, on the basis of the information provided by our model, try to set up an intelligible scheme in order to understand the "mantle-bouncing" effect recorded above. In this sense we call the attention to Figure 6*b*. Observe the increasing, before the bouncing, of the nonadiabatic index at the outer regions and the subsequent "relaxation" after bouncing. This stiffening-softening of the outer mantle, clearly leading to an expansion, is preceded by the appearance of a minimum of the nonadiabatic index, just at the outer side of the shock. This will lead to the "breaking" of the sphere (see Fig. 8) at the shock front. As a result of these two effects (stiffening-softening plus "breaking") we shall have a compact, homogeneous

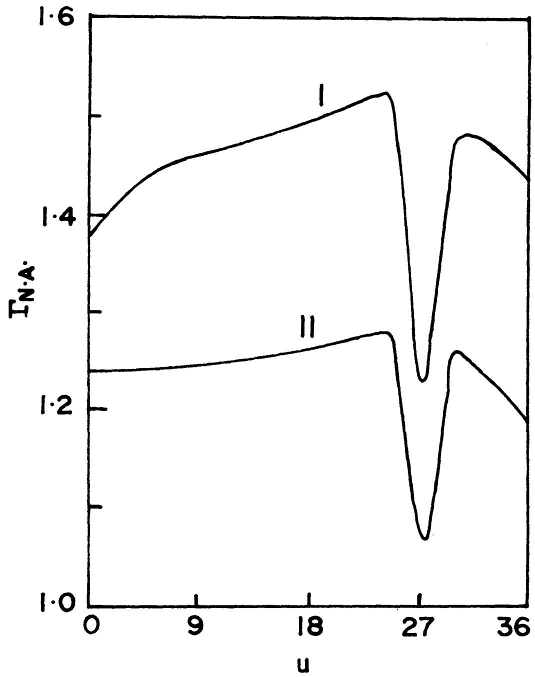

FIG. 7.—Evolution of the nonadiabatic index at both sides of the shock

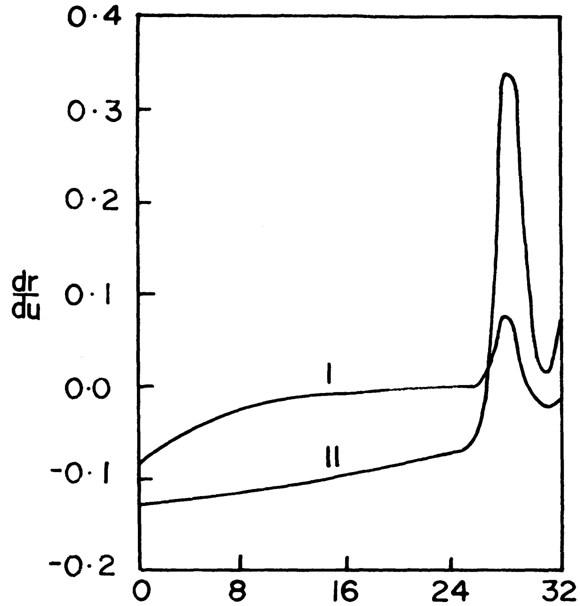

FIG. 8.—Evolution of the matter velocity at both sides of the shock

"remnant" and an ever-expanding mantle. This picture is very similar to the one described before (Herrera & Núñez 1987, 1990). However, in those models the streaming-out limit was used throughout. As a result of this the "breaking" of the sphere occurred, in those models, not at the shock front, but somewhere farther from the center. This difference brings out further the higher "efficiency" of the diffusion limit as compared with the streaming-out approximation (Barreto, Herrera, & Santos 1989).

Thus in the context of a consistent general relativistic framework and adopting simple and highly idealized but not extremely unphysical equations of state, we were able to set up a bouncing mechanism not involving propagation of a strong shock wave reaching the boundary surface. However, the extent to which this result could be of any use is clearly restricted by the main limitations of our approach, namely:

1. The two "seed" equations of state used at either side of the shock, although not completely deprived of physical meaning, have not been obtained from a realistic treatment of ultradense matter. Both the microphysics of the fluid and the link between the specific luminosity profile and the particular regime of radiation transport are absent.
2. The assumption of pure diffusion is too strong as to be applied to the collapse of a real massive star.
3. The particular profile of the stiffness exhibited in Figure 6b is not related in any way to the microphysics of the fluid.
4. There is no neutrino physics associated with the heat flow vector q^μ.

APPENDIX

EQUATIONS FOR THE DISCONTINUITY SURFACE

From the Rankine-Hugoniot condition (28) we obtain

$$\Omega_s^2 + a\Omega_s + b = 0 \qquad \text{(A1)}$$

where

$$a = \frac{P_s + \tilde{P}_s}{P_s + \rho_s} + \frac{1}{2}$$

$$b = \frac{Q_s + \tilde{P}_s + \tilde{\rho}_s}{2(P_s + \rho_s)}$$

with

$$Q_s = \frac{q_s(2\Omega - 1)^{3/2}}{4\pi\Omega_s C^2 G^{1/2}}$$

$s =$ I, II indicating the region they belong to. The field equation (18) evaluated at both sides of the shock leads to the quartic equation:

$$\Omega_s^4 - 2\Omega_s^3 + \alpha\Omega_s^2 + \beta\Omega_s + \pi = 0 , \quad \text{(A2)}$$

where

$$\alpha = \frac{9\tilde{p}_s + 5\tilde{P}_s}{4(\tilde{p}_s + \tilde{P}_s)} + \lambda ,$$

$$\beta = - \frac{5\tilde{p}_s + \tilde{P}_s}{4(\tilde{p}_s + \tilde{P}_s)} - \lambda ,$$

$$\pi = - \frac{\tilde{p}_s}{4(\tilde{p}_s + \tilde{P}_s)} + \frac{1}{4}\lambda ,$$

and

$$\lambda = Q_s - \left(\Psi\tilde{p}_s - \frac{\dot{m}_s e^{-2\beta_s}}{4\pi C^2 G}\right) ,$$

with the property

$$\alpha + \beta = 1 ,$$

it can be shown that equations (A2) have solutions

$$\Omega_s = \pm \sqrt{\left[\frac{3}{4} - \frac{\alpha}{2} \pm \left(\frac{\alpha}{2} - \frac{3}{4}\right)\right]} + \frac{1}{2} . \quad \text{(A3)}$$

Two of these solutions correspond to discontinuities that travel with the light velocity. Then, for physical models ($-1 < \omega < 1$, $\rho > P, \rho > 0$) the only solution results

$$\Omega_s = \sqrt{\frac{3}{2} - \alpha} + \frac{1}{2} , \quad \text{(A4)}$$

substituting equation (A4) in equation (A1) and evaluating at both sides of the shock, we obtain at $r = c + o$

$$\chi\dot{G}^3 + \Phi\dot{G}^2 + \eta\dot{G} + \Sigma = 0 \quad \text{(A5)}$$

which is solved algebraically in G. At $r = c - o$, we have

$$\dot{D} = \frac{\Upsilon}{\Theta} \quad \text{(A6)}$$

where all quantities are listed below:

$\chi = -OR$

$\Phi = (1.75 - P)R^2 - O^2 - R(1 + Q)O$

$\eta = 2(1.75 - P)(1.5 - P)(1 + Q) + 2(1.75 - P)O - (1 + Q)O$

$\Sigma = (1 + Q)^2(1.5 - P) - (1.75 - P)^2$

$\Upsilon = U - [1.75 + (W + 1)(1.5 - U)^{1/2} + 0.5(W - V)]$

$\Theta = 0.5(V - W) - V(1.5 - U)^{1/2}$

$O = -1/(8\pi C\,\Delta e_3)$

$P = [q_{\text{IIc}}/(4\pi C^2) + (2.25 - Y)e_1 + 1.25e_2 + (1 - G)\dot{C}/(8\pi C^2 G\Delta)]/e_3$

$R = b_1/(4\pi\,\Delta e_3)$

$Q = -[-b_2/(4\pi\Delta)] + G(2b_3 + 4b_4^2 - b_4/C)/(8\pi) + [3b_4(1 - 2m_1) - m_2]/[8\pi C + 1.5e_1 + 0.5e_2]/e_3$

$U = [0.25(9e_4 + 5e_5) + q_{\text{Ic}}/(4\pi C^2) - \Psi e_4 + m_3]/e_6$

$V = b_5/(4\pi\,\Delta e_6)$

$W = -[-b_6/(4\pi\Delta)] + G(2b_7 + 4b_8^2 - b_8/C)/(8\pi) + [3b_8(1 - 2m_4) - m_5]/[8\pi C + 1.5e_4 + 0.5e_5]/e_6$

$e_1 = \tilde{\rho}_{\text{IIc}} = 3H/C^2$

$e_2 = P_{\text{IIc}} = e_1(1 - 9KC)/[3(1 - KC)]$

$e_3 = \tilde{\rho}_{IIc} + \tilde{P}_{IIc}$

$e_4 = \tilde{\rho}_{Ic} = 3(1 - G)/(8\pi C^2)$

$e_5 = -e_4(D + 2/3)/D = \tilde{P}_{Ic}$

$e_6 = \tilde{\rho}_{Ic} + \tilde{P}_{Ic}$

$\Delta = e^{2\beta c} = \zeta^2/\xi^2$

$\zeta = \Phi^{\Lambda}$

$\xi = \theta^{\Gamma}$

$\phi = (\gamma + \delta C)/(\gamma + \delta A)$

$\Lambda = -8\pi H(\delta + 3K\gamma)/[\delta(\gamma K + \delta)]$

$\theta = (1 - KC)/(1 - KA)$

$\Gamma = 16\pi H/(\gamma K - \delta)$

$\Psi = -\{3D(1 - KC)[(1 - G)(3A - 2C) - A(1 - F)]/[4D(1 - 3KC)A(1 - F)]\}$

$b_1 = b_4[H_a/H - (\gamma_a + \delta_a C)]/(\gamma + \delta C) - 2K_a C/[(1 - KC)(1 - 3KC)]$

$b_2 = b_4[H_b/H - (\gamma_b + \delta_b C)]/(\gamma + \delta C) - 2K_b C/[(1 - KC)(1 - 3KC)]$

$b_3 = -b_4(\delta/(\gamma + \delta C) + 2\{(4\Omega - 3)/[3A(4\Omega - 1)]\}/[(1 - KC)(1 - 3KC)])$

$b_4 = 8\pi H(1 - 3KC)/(1 - KC)(\gamma + \delta C)$

$b_5 = -b_8/D$

$b_6 = b_8[(f_a \dot{G} + f_b)/f - \dot{G}/G - 2(1 - G)\dot{C}/(CG) - (1 - D)(1 - G)\dot{C}/(GCD)]b_8$

$b_7 = b_8[D(2 - G) + (D + 1)(1 - G)/(CGD)]$

$b_8 = -4\pi Cf/(3GD)$

$m_1 = 12\pi H = \tilde{m}_{II1,c}$

$m_2 = 0 = \tilde{m}_{11,c}$

$m_3 = 0.5[\dot{C}(1 - G) - \dot{G}C]/(4\pi C^2 6D) = \dot{m}_c$

$m_4 = 1.5(1 - G) = \tilde{m}_{I1,c}$

$m_5 = 3(1 - G)/C = \tilde{m}_{I11,c}$

$H = [A(1 - F) - C(1 - G)]/[24\pi(A - C)]$

$H_a = C/[24\pi(A - C)]$

$H_b = \{[\dot{A}(1 - F) - A\dot{F} - \dot{C}(1 - G)](A - C) - [A(1 - F) - C(1 - G)](\dot{A} - \dot{C})\}/[24\pi(A - C)^2]$

$\gamma = 8\pi(3\pi CH - C^3 f/3)$

$\gamma_a = 24\pi CH_a - 8\pi f_a C^3/3$

$\gamma_b = 24\pi(CH + CH_b) - 8\pi C^2(3\dot{C}f + Cf_b)/3$

$\delta = 1 - 24\pi H$

$\delta_a = -24\pi H_a$

$\delta_b = -24\pi H_b$

$f = 3(1 - G)/(8\pi C^2)$

$f_a = -3/(8\pi C^2)$

$f_b = -3(1 - G)\dot{C}/(4\pi C^3)$

$\kappa_a = K_a \Omega_a$

$\kappa_b = K_a \Omega_b + K_b$

$K_a = 8/[3A(4\Omega - 1)^2]$

$K_b = -\dot{A}(4\Omega - 3)/[3A^2(4\Omega - 1)]$

$$\Omega_a = \Omega H_a/H$$

$$\Omega_b = \Omega H_b/H + \Omega\{-2\dot{A}/A - \dot{F}/F - (\Omega - 1)[-2F\Omega/A + 12\pi H(3\Omega - 1)/(A\Omega) - 0.5(3 + F)/A] + \Omega q/[6H(2\Omega - 1)\pi A] - \Omega[0.5(1 - F) - 12\pi H(\Omega - 1)/\Omega - 3F/(8\Omega)]/A\}$$

$$K = (4\Omega - 3)/[3A(4\Omega - 1)]$$

$$A = a(u)/m(0)\,, \qquad F = 1 - 2M/A\,, \qquad M = m_a/m(0)$$

$$\Omega = 1/(1 - \omega_a)\,, \qquad C = c(u)/m(0)\,, \qquad G = 1 - 2M/C\,, \qquad M = m_c/m(0)$$

$$D = gG^{1/2} - 1\,, \qquad g \text{ arbitrary}$$

and finally q, q_{IIc}, and q_{Ic} are the heat flux given as Gaussian-like at $r = a(u)$, $r = c(u) + 0$, and $r = c(u) - 0$, respectively.

REFERENCES

Barreto, W., Herrera, L., & Santos, N. 1989, ApJ, 344, 158
———. 1990, Ap&SS, submitted
Bondi, H. 1964, Proc. Roy. Soc. London A, 281, 39
Bondi, H., Van der Burg, M. G. Y., & Metzner, A. W. K. 1962, Proc. Roy. Soc. London A, 269, 21
Herrera, L., Jiménez, J., Núñez, L., & Esculpi, M. 1987, Phys. Letters, 124A, 248
Herrera, L., & Núñez, L. 1987, ApJ, 319, 868
———. 1990, ApJ, 364, 212
Kazanas, D. 1978, ApJ, 222, L109
Lake, K. 1987, in 5th Brazilian School of Cosmology and Gravitation, ed. M. Novello (Singapore: World Scientific), p. 6
Lattimer, J. M. 1988, Nucl. Phys., A478, 199c
Lattimer, J., Burrows, A., & Yahil, A. 1985, ApJ, 288, 644
Mihalas, D., & Mihalas, B. 1984, Foundations of Radiation Hydrodynamics (Oxford: Oxford University Press)
Santos, N. O. 1985, MNRAS, 216, 403
Taub, A. H. 1948, Phys. Rev., 74, 328
———. 1983, in Proceedings of the Third Marcel Grossman Meeting on General Relativity, ed. H. Ning (North-Holland: Science Press), p. 165
Tolman, R. C. 1939, Phys. Rev., 55, 364
Yahil, A. 1983, ApJ, 265, 1047